On the sparsity of MRD codes

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joint work with Eimear Byrne

Density of MDS codes

A k-dimensional block code $\mathscr{C} \leq \mathbb{F}_q^n$ is **MDS** if $d_{\mathrm{H}}(\mathscr{C}) = n-k+1$.

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In other words...

Theorem (Folklore)

Let $n \geq k \geq 1$ be integers. We have

$$\lim_{q \to +\infty} \frac{\# \text{ of } k\text{-dim MDS codes in } \mathbb{F}_q^n}{\# \text{ of } k\text{-dim codes in } \mathbb{F}_q^n} = 1.$$

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We say that MDS codes are **dense** within the set of k-dimensional codes in $\mathbb{F}_q^n$.

The notion of density

Definition

Let $S \subseteq \mathbb{N}$ be an infinite set. Let $(\mathscr{F}_s \mid s \in S)$ be a sequence of finite non-empty sets indexed by S, and let $(\mathscr{F}'_s \mid s \in S)$ be a sequence of sets with $\mathscr{F}'_s \subseteq \mathscr{F}_s$ for all $s \in S$.

The **density function** $S \to \mathbb{Q}$ of $\mathscr{F}'_s$ in $\mathscr{F}_s$ is $s \mapsto |\mathscr{F}'_s|/|\mathscr{F}_s|$.

If
$$\lim_{s \to +\infty} |\mathscr{F}'_s|/|\mathscr{F}_s| = \delta,$$
then $\mathscr{F}'_s$ has **density** δ in $\mathscr{F}_s$.

- $\delta = 1$: $\mathscr{F}'_s$ is **dense** in $\mathscr{F}_s$
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EXAMPLE: $S = \mathbb{N}_{\geq 1}$ $\quad \mathscr{F}_s = \{n \in \mathbb{N} \mid 1 \leq n \leq s\}$ $\quad \mathscr{F}'_s = \{p \in \mathbb{N} \mid p \leq s,\ p \text{ prime}\}$.

Then: $|\mathscr{F}'_s|/|\mathscr{F}_s| \to 0,$ $\quad |\mathscr{F}'_s|/|\mathscr{F}_s| \sim 1/\log(s)$

(Hadamard, de la Vallée-Poussin, 1896)

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EXAMPLE: $S = \mathbb{N}$ $\quad \mathscr{F}_s = \{n \in \mathbb{N} \mid n \leq s\}$ $\quad \mathscr{F}'_s = \{n \in \mathbb{N} \mid n \text{ is even}\}$.

Then:
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The even numbers have density 1/2 within the natural numbers.

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Density of MDS codes

Remark

Let $G \in \mathbb{F}_q^{k \times n}$ is a rank k matrix in reduced row-echelon form. TFAE:

1. the rows of G generate a k-dimensional MDS code;
2. all the $k \times k$ minors of G are non-zero (in particular, $\text{piv}(G) = \{1, ..., k\}$).

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Consider a matrix of the form $G = (I_k \mid Y)$, where Y is a $k \times (n-k)$ matrix of independent variables $(z_i \mid 1 \leq i \leq N)$ and $N = k(n-k)$.

$$\text{e.g.} \qquad \begin{pmatrix} 1 & 0 & z_1 & z_2 & z_3 & z_4 \\ 0 & 1 & z_5 & z_6 & z_7 & z_8 \end{pmatrix} \qquad N = 8$$

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Let $p_1, \dots, p_M \in \mathbb{F}_q[z_1, \dots, z_N]$ be the maximal minors of G, where $M = \binom{n}{k}$. The MDS codes correspond to the vectors $(\alpha_1, \dots, \alpha_N) \in \mathbb{F}_q^N$ such that

$$(p_1 p_2 \cdots p_M)(\alpha_1, \dots, \alpha_N) \neq 0.$$

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The k-dimensional MDS codes in $\mathbb{F}_q^n$ correspond to the non-zeros $(\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ of a polynomial $p := p_1 p_2 \cdots p_M \in \mathbb{F}_q[z_1, \ldots, z_N]$.

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Density problems in Coding Theory

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We study "density questions" in coding theory in:

E. Byrne, A. Ravagnani
Partition-Balanced Families of Codes and Asymptotic Enumeration in Coding Theory
arXiv 1805.02049

Density problems in coding theory

We study density problems in general:

- Ambient space: Hamming space, vector rk-metric space, matrix rk-metric space
- Various properties related to: minimum distance, covering radius, maximality

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Idea

Look at **families** of codes that exhibit regularity properties with respect to partitions of the ambient space $X \in \{\mathbb{F}_q^n,\ \mathbb{F}_{q^m}^n,\ \mathbb{F}_q^{n\times m}\}$.

Definition

Let $\mathscr{P} = \{P_1, P_2, ..., P_\ell\}$ be a partition of X. A family $\mathscr{F}$ of codes in X is $\mathscr{P}$-**balanced** if for all $x \in X$ the number

$$|\{\mathscr{C} \in \mathscr{F} \mid x \in \mathscr{C}\}|$$

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We use $\mathscr{P}$-balanced families to estimate the number of codes with a certain properties.

What is the density of optimal/non-optimal codes?

Hamming space

- $X = \mathbb{F}_q^n, \quad d_H(x,y) = |\{i \mid x_i \neq y_i\}|$
- Code: $\mathbb{F}_q$-subspace $\mathscr{C} \leq \mathbb{F}_q^n$
- Bound: a code $\mathscr{C} \leq \mathbb{F}_q^n$ has $\dim(\mathscr{C}) \leq n - d_H(\mathscr{C}) + 1$
- Codes meeting the bound: MDS codes (optimal)

Vector rank-metric space

- $X = \mathbb{F}_{q^m}^n$ with $m \geq n, \quad d_{rk}(x,y) = \dim_{\mathbb{F}_q} \mathrm{span}\{x_1 - y_1, \ldots, x_n - y_n\}$
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Matrix rank-metric space

- $X = \mathbb{F}_q^{n \times m}$ with $m \geq n, \quad d_{rk}(x,y) = \mathrm{rk}(X - Y)$
- Code: $\mathbb{F}_q$-subspace $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$
- Bound: a code $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$ has $\dim(\mathscr{C}) \leq m(n - d_{rk}(\mathscr{C}) + 1)$
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MRD vector rk-metric codes

Using the Schwartz-Zippel lemma:

Theorem (Neri-Trautmann-Randrianarisoa-Rosenthal, 2017)

$$\frac{\#\text{ of } k\text{-dim MRD codes in } \mathbb{F}_{q^m}^n}{\#\text{ of } k\text{-dim codes in } \mathbb{F}_{q^m}^n} \geq q^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m}^{-1} \left(1 - \sum_{r=0}^{k} \begin{bmatrix} k \\ k-r \end{bmatrix}_q \begin{bmatrix} n-k \\ r \end{bmatrix}_q q^{r^2} q^{-m}\right)$$

$$\rightarrow 1 \text{ as } m \rightarrow +\infty$$

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We can improve this bound as follows:

Theorem (Byrne-R.)

$$\frac{\#\text{ of } k\text{-dim MRD codes in } \mathbb{F}_{q^m}^n}{\#\text{ of } k\text{-dim codes in } \mathbb{F}_{q^m}^n} \geq 1 - \frac{q^{mk}-1}{(q^m-1)(q^{mn}-1)} \left(-1 + \sum_{i=0}^{d-1} \begin{bmatrix} n \\ i \end{bmatrix}_q \prod_{j=0}^{i-1} (q^m - q^j) \right)$$

MRD matrix rk-metric codes

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Theorem (Byrne-R.)

Let $m \geq n \geq 2$ and let $1 \leq k \leq mn-1$ be integers.

- If m does not divide k, then there is no k-dimensional MRD code $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$.
- If m divides k, then

$$\frac{\#\text{ of } k\text{-dim non-MRD codes in } \mathbb{F}_q^{n\times m}}{\#\text{ of } k\text{-dim codes in } \mathbb{F}_q^{n\times m}} \geq$$

$$q \begin{bmatrix} mn \\ k \end{bmatrix}^{-1} \left(\sum_{h=1}^{m(n-k)} \begin{bmatrix} t \\ h \end{bmatrix} \sum_{s=h}^{m(n-k)} \begin{bmatrix} m(n-k)-h \\ s-h \end{bmatrix} \begin{bmatrix} mn-s \\ mn-k \end{bmatrix} (-1)^{s-h} q^{\binom{s-h}{2}} \right) \cdot$$

$$\cdot \left(1 - \frac{\left(q^k-1\right)\left(q^{mn-k}-1\right)}{2\left(q^{mn}-q^{mn-k}\right)} \right).$$

This quantity goes to $1/2$ as $q \to +\infty$ and to $1/2(q/(q-1)-(q-1)^2)$ as $m \to +\infty$.

Non-density of MRD matrix codes

Corollary (Byrne-R.)

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Matrix MRD codes are NOT dense.

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Non-density for $q \to +\infty$ was also shown by Antrobus/Gluesing-Luerssen with different methods.

Other results

We can study:

- Density of codes that are **optimal** (MDS, MRD, MRD)
- Density of codes of bounded **minimum distance**
- Density of codes that meet the *redundancy bound* for their **covering radius**
- Density of matrix codes that meet the *initial set bound* for their covering radius
- Density of optimal codes within **maximal** codes (with respect to inclusion)
- **Average parameters** of codes (e.g., average weight distribution)
- ...

Covering radius of rk-metric codes

Theorem (Byrne, R.)

Let k be an integer with $0 \leq k \leq nm$. Denote by $\mathscr{F}$ the family of rank metric codes in $\mathbb{F}_q^{n\times m}$ of dimension k. Define $\rho_k := n - \lfloor k/m \rfloor$, and let $\mathscr{F}' := \{\mathscr{C} \in \mathscr{F} \mid \rho^{\mathrm{rk}}(\mathscr{C}) = \rho_k\}$.

Recall: $\rho^{\mathrm{rk}}(\mathscr{C}) = \min\{i \mid \text{for all } N \in \mathbb{F}_q^{n\times m} \text{ there exists } M \in \mathscr{C} \text{ with } \mathrm{rk}(M,N) \leq i\}$.

We have

$$\lim_{q\to\infty} \frac{|\mathscr{F}'|}{|\mathscr{F}|} = 1, \qquad \text{provided that } k < (m-n+\lfloor k/m \rfloor +1)(\lfloor k/m \rfloor +1).$$

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Remark

Not all rk-metric codes have covering radius $n - \lfloor k/m \rfloor$.

Theorem (Byrne, R.)

Let k be an integer with $0 \leq k \leq nm$. Denote by $\mathscr{F}$ the family of rank metric codes in $\mathbb{F}_q^{n\times m}$ of dimension k. Define $\rho_k := n - \lfloor k/m \rfloor$, and let $\mathscr{F}' := \{\mathscr{C} \in \mathscr{F} \mid \rho^{\mathrm{rk}}(\mathscr{C}) = \rho_k\}$.

Recall: $\rho^{\mathrm{rk}}(\mathscr{C}) = \min\{i \mid \text{for all } N \in \mathbb{F}_q^{n\times m} \text{ there exists } M \in \mathscr{C} \text{ with } \mathrm{rk}(M,N) \leq i\}$.

We have

$$\lim_{q\to\infty} \frac{|\mathscr{F}'|}{|\mathscr{F}|} = 1, \qquad \text{provided that } k < (m-n+\lfloor k/m \rfloor+1)(\lfloor k/m \rfloor+1).$$

Remark

Not all rk-metric codes have covering radius $n - \lfloor k/m \rfloor$.

Thank you very much!