The Covering Radius of Rank-Metric Codes

Alberto Ravagnani

– University College Dublin –

Algebraic Coding Theory for Networks, Storage, and Security
Schloss Dagstuhl, Dec. 2018

Rank-metric codes

Definition

A (**rank-metric**) **code** is a non-empty subset $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$. We assume $n \le m$ w.l.o.g.

The (**rank**) **distance** between matrices $M, N \in \mathbb{F}_q^{n\times m}$ is $\mathrm{rk}(M-N)$.

If $|\mathscr{C}| \geq 2$, then the **minimum distance** of $\mathscr{C}$ is

$$d(\mathscr{C}) := \min\{\mathrm{rk}(M-N) \mid M, N \in \mathscr{C}, M \neq N\}.$$

Rank-metric codes

Definition

A (**rank-metric**) **code** is a non-empty subset $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$. We assume $n \leq m$ w.l.o.g.

The (**rank**) **distance** between matrices $M, N \in \mathbb{F}_q^{n \times m}$ is $\mathrm{rk}(M - N)$.

If $|\mathscr{C}| \geq 2$, then the **minimum distance** of $\mathscr{C}$ is

$$d(\mathscr{C}) := \min\{\mathrm{rk}(M - N) \mid M, N \in \mathscr{C}, M \neq N\}.$$

We say that $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ is **linear** if it is an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{n \times m}$. In this case the **dual** of $\mathscr{C}$ is the linear code

$$\mathscr{C}^\perp := \{N \in \mathbb{F}_q^{n \times m} \mid \mathrm{Tr}(MN^t) = 0 \text{ for all } M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{n \times m}.$$

Rank-metric codes

Definition

A (**rank-metric**) **code** is a non-empty subset $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$. We assume $n \leq m$ w.l.o.g.

The (**rank**) **distance** between matrices $M, N \in \mathbb{F}_q^{n\times m}$ is $\mathrm{rk}(M-N)$.

If $|\mathscr{C}| \geq 2$, then the **minimum distance** of $\mathscr{C}$ is

$$d(\mathscr{C}) := \min\{\mathrm{rk}(M-N) \mid M, N \in \mathscr{C}, M \neq N\}.$$

We say that $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ is **linear** if it is an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{n\times m}$. In this case the **dual** of $\mathscr{C}$ is the linear code

$$\mathscr{C}^\perp := \{N \in \mathbb{F}_q^{n\times m} \mid \mathrm{Tr}(MN^t) = 0 \text{ for all } M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{n\times m}.$$

- Introduced by Delsarte for combinatorial interest via association schemes.
- Introduced independently by Gabidulin and Roth.
- Re-discovered by Kötter-Kschischang-Silva and applied to linear network coding.

Rank-metric codes

Definition

A (**rank-metric**) **code** is a non-empty subset $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$. We assume $n \leq m$ w.l.o.g.

The (**rank**) **distance** between matrices $M, N \in \mathbb{F}_q^{n\times m}$ is $\mathrm{rk}(M-N)$.

If $|\mathscr{C}| \geq 2$, then the **minimum distance** of $\mathscr{C}$ is

$$d(\mathscr{C}) := \min\{\mathrm{rk}(M-N) \mid M, N \in \mathscr{C}, M \neq N\}.$$

We say that $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ is **linear** if it is an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{n\times m}$. In this case the **dual** of $\mathscr{C}$ is the linear code

$$\mathscr{C}^\perp := \{N \in \mathbb{F}_q^{n\times m} \mid \mathrm{Tr}(MN^t) = 0 \text{ for all } M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{n\times m}.$$

- Introduced by Delsarte for combinatorial interest via association schemes.
- Introduced independently by Gabidulin and Roth.
- Re-discovered by Kötter-Kschischang-Silva and applied to linear network coding.

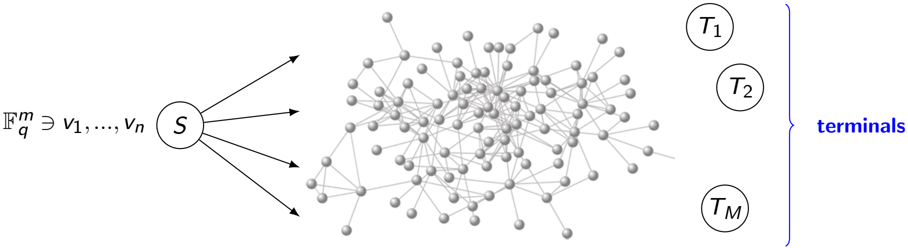
What is linear network coding?

What is network coding about?

Network coding: data transmission over networks.

What is network coding about?

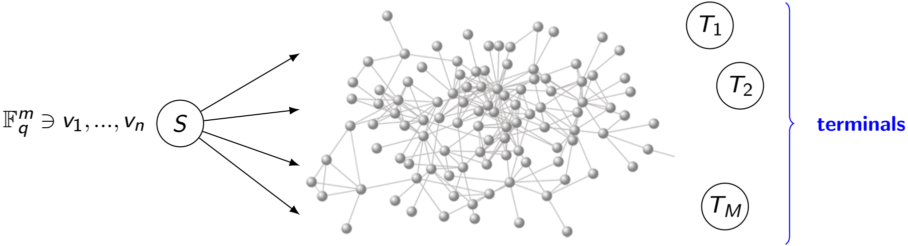
Network coding: data transmission over networks.



- One source S attempts to transmit messages $v_1, \ldots, v_n \in \mathbb{F}_q^m$.
- The terminals demand **all** the messages (multicast).

What is network coding about?

Network coding: data transmission over networks.

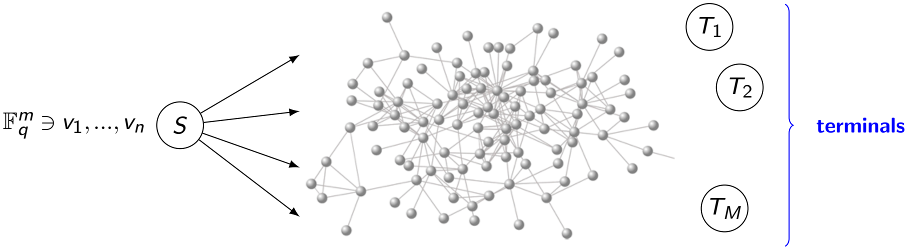


- One source S attempts to transmit messages $v_1, \ldots, v_n \in \mathbb{F}_q^m$.
- The terminals demand **all** the messages (multicast).

What should the nodes do?

What is network coding about?

Network coding: data transmission over networks.



- One source S attempts to transmit messages $v_1, \dots, v_n \in \mathbb{F}_q^m$.
- The terminals demand **all** the messages (multicast).

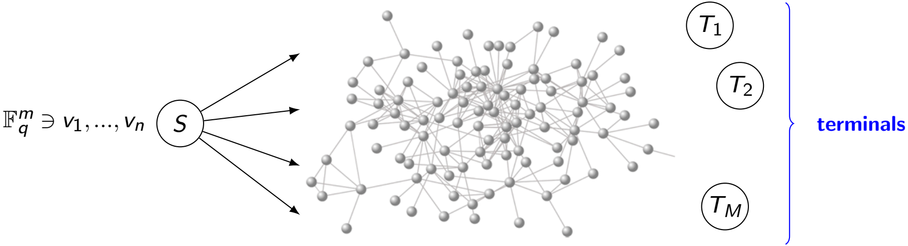
What should the nodes do?

Goal

Maximize the messages that are transmitted to **all** terminals per channel use (**rate**).

What is network coding about?

Network coding: data transmission over networks.



- One source S attempts to transmit messages $v_1, \dots, v_n \in \mathbb{F}_q^m$.
- The terminals demand **all** the messages (multicast).

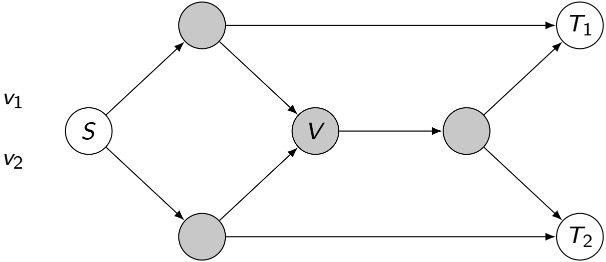
What should the nodes do?

Goal

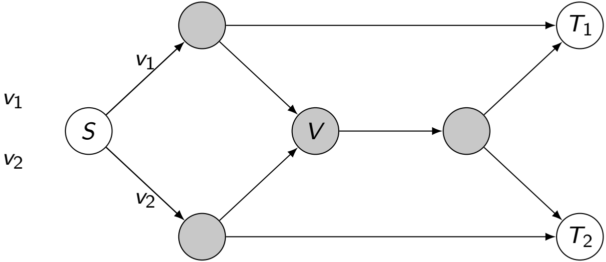
Maximize the messages that are transmitted to **all** terminals per channel use (**rate**).

IDEA (Ahlswede-Cai-Li-Yeung 2000): allow the nodes to recombine packets.

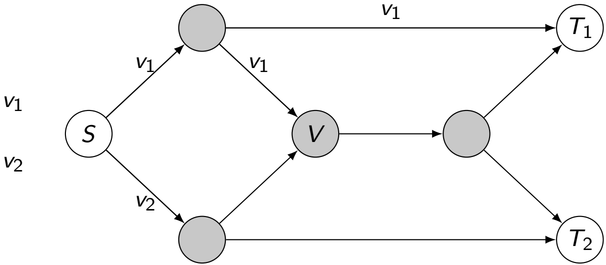
The "Butterfly" network



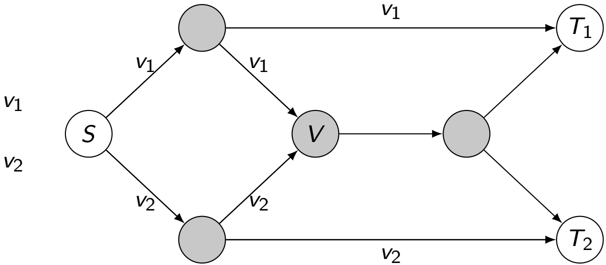
The "Butterfly" network



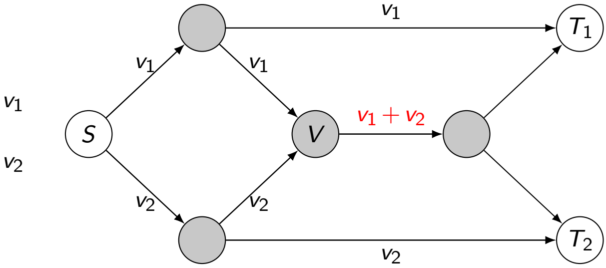
The “Butterfly” network



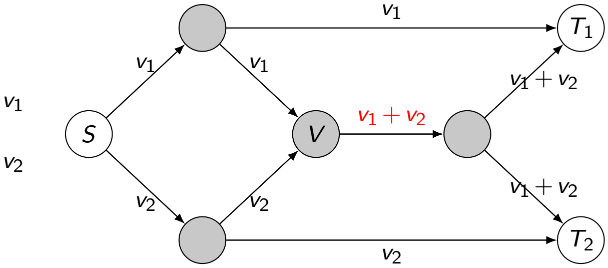
The “Butterfly” network



The "Butterfly" network



The “Butterfly” network



This strategy is better than routing.

Error correction in networks

Error correction in networks

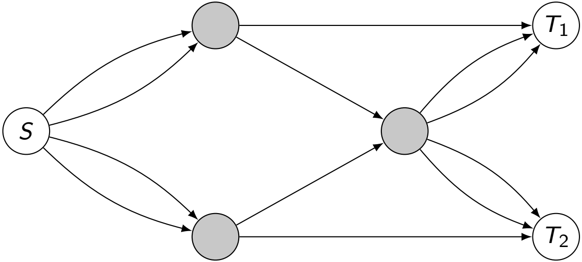
The model

One adversary can change the value of up to t edges (t is the adversarial *strength*).

Error correction in networks

The model

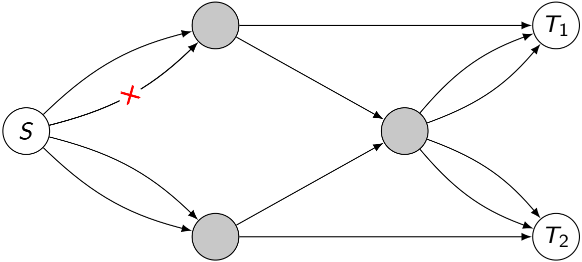
One adversary can change the value of up to t edges (t is the adversarial *strength*).



Error correction in networks

The model

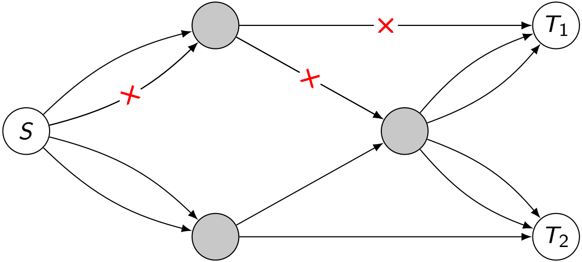
One adversary can change the value of up to t edges (t is the adversarial *strength*).



Error correction in networks

The model

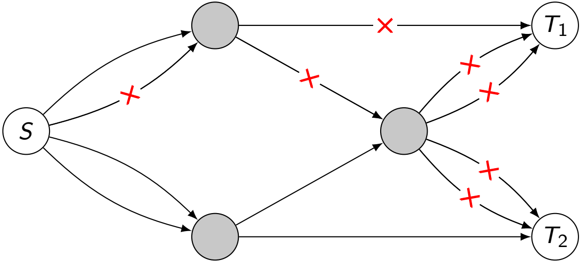
One adversary can change the value of up to t edges (t is the adversarial *strength*).



Error correction in networks

The model

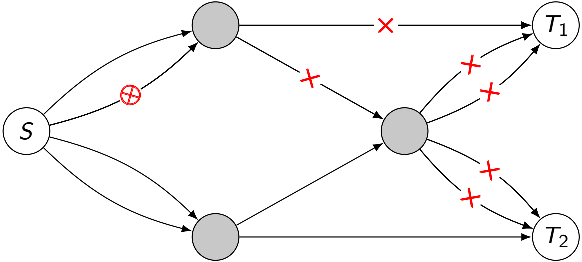
One adversary can change the value of up to t edges (t is the adversarial *strength*).



Error correction in networks

The model

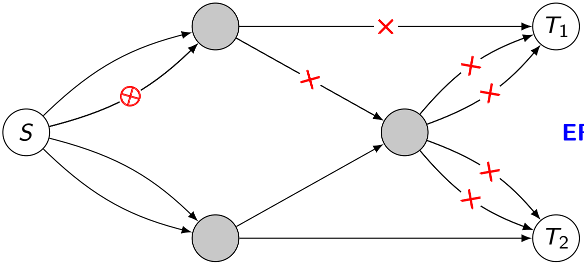
One adversary can change the value of up to t edges (t is the adversarial *strength*).



Error correction in networks

The model

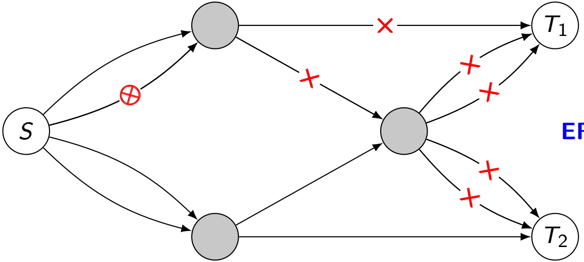
One adversary can change the value of up to t edges (t is the adversarial *strength*).



ERROR AMPLIFICATION

The model

One adversary can change the value of up to t edges (t is the adversarial *strength*).

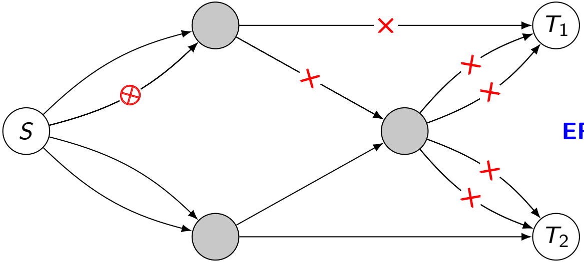


ERROR AMPLIFICATION

Natural approach: number of corrupted edges as a measure for the "disaster".

The model

One adversary can change the value of up to t edges (t is the adversarial *strength*).



ERROR AMPLIFICATION

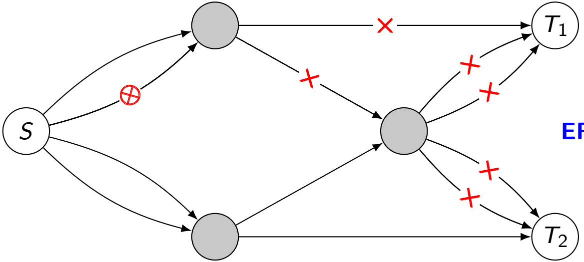
Natural approach: number of corrupted edges as a measure for the "disaster".

Convenient approach: use rank-metric codes.

Error correction in networks

The model

One adversary can change the value of up to t edges (t is the adversarial *strength*).



ERROR AMPLIFICATION

Natural approach: number of corrupted edges as a measure for the "disaster".

Convenient approach: use rank-metric codes.

According to the rank metric, **errors** propage but **do not amplify**.

Covering Radius

Back to the mathematical theory of rank-metric codes...

Byrne, R., *Covering radius of matrix codes endowed with the rank metric*. SIAM J. Discrete Math.

Definition

The **covering radius** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ is the integer

$$\rho(\mathscr{C}) := \min\{i \in \mathbb{N} \mid \text{for all } X \in \mathbb{F}_q^{n \times m} \text{ there exists } M \in \mathscr{C} \text{ with } d(X, M) \leq i\}$$

Covering Radius

Back to the mathematical theory of rank-metric codes...

Byrne, R., *Covering radius of matrix codes endowed with the rank metric*.
SIAM J. Discrete Math.

Definition

The **covering radius** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ is the integer

$$\rho(\mathscr{C}) := \min\{i \in \mathbb{N} \mid \text{for all } X \in \mathbb{F}_q^{n \times m} \text{ there exists } M \in \mathscr{C} \text{ with } d(X, M) \leq i\}$$

This the rank-analogue of the covering radius of a code $C \subseteq \mathbb{F}_q^n$ endowed with the Hamming metric.

Covering Radius

Back to the mathematical theory of rank-metric codes...

Byrne, R., *Covering radius of matrix codes endowed with the rank metric*.
SIAM J. Discrete Math.

Definition

The **covering radius** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ is the integer

$$\rho(\mathscr{C}) := \min\{i \in \mathbb{N} \mid \text{for all } X \in \mathbb{F}_q^{n \times m} \text{ there exists } M \in \mathscr{C} \text{ with } d(X, M) \leq i\}$$

This the rank-analogue of the covering radius of a code $C \subseteq \mathbb{F}_q^n$ endowed with the Hamming metric.

$\rho(\mathscr{C})$ is the minimum value r such that the union of the spheres of radius r about the codeword cover the ambient space.

Covering radius of vector rank-metric codes ($\mathbb{F}_{q^m}$-linear) studied by Gadouleau-Yan:

Gadouleau, Yan *Packing and Covering Properties of Rank Metric Codes*.
IEEE Transactions Inf. Th.

First properties of the covering radius

Lemma

Let $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ be a code. The following hold.

1. $0 \leq \rho(\mathscr{C}) \leq n$. Moreover, $\rho(\mathscr{C}) = 0$ if and only if $\mathscr{C} = \mathbb{F}_q^{n\times m}$.
2. If $\mathscr{D} \subseteq \mathbb{F}_q^{n\times m}$ is a code with $\mathscr{C} \subseteq \mathscr{D}$, then $\rho(\mathscr{C}) \geq \rho(\mathscr{D})$.
3. If $\mathscr{D} \subseteq \mathbb{F}_q^{n\times m}$ is a code with $\mathscr{C} \subsetneq \mathscr{D}$, then $\rho(\mathscr{C}) \geq d(\mathscr{D})$.

First properties of the covering radius

Lemma

Let $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ be a code. The following hold.

1. $0 \leq \rho(\mathscr{C}) \leq n$. Moreover, $\rho(\mathscr{C}) = 0$ if and only if $\mathscr{C} = \mathbb{F}_q^{n\times m}$.
2. If $\mathscr{D} \subseteq \mathbb{F}_q^{n\times m}$ is a code with $\mathscr{C} \subseteq \mathscr{D}$, then $\rho(\mathscr{C}) \geq \rho(\mathscr{D})$.
3. If $\mathscr{D} \subseteq \mathbb{F}_q^{n\times m}$ is a code with $\mathscr{C} \subsetneq \mathscr{D}$, then $\rho(\mathscr{C}) \geq d(\mathscr{D})$.

A code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ is **maximal** if $|\mathscr{C}| = 1$ or $|\mathscr{C}| \geq 2$ and there is no code $\mathscr{D} \subseteq \mathbb{F}_q^{n\times m}$ with $\mathscr{D} \supsetneq \mathscr{C}$ and $d(\mathscr{D}) = d(\mathscr{C})$. In particular, $\mathbb{F}_q^{n\times m}$ is maximal.

Proposition

A code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ with $|\mathscr{C}| \geq 2$ is maximal if and only if $\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1$.

Maximality

We introduce a parameter that measures the maximality of a code.

Definition

The **maximality degree** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ with $|\mathscr{C}| \geq 2$ is the integer defined by

$$\mu(\mathscr{C}) := \begin{cases} \min\{d(\mathscr{C}) - d(\mathscr{D}) \mid \mathscr{D} \subseteq \mathbb{F}_q^{n \times m} \text{ is a code with } \mathscr{D} \supsetneq \mathscr{C}\} & \text{if } \mathscr{C} \subsetneq \mathbb{F}_q^{n \times m}, \\ 1 & \text{if } \mathscr{C} = \mathbb{F}_q^{n \times m}. \end{cases}$$

Maximality

We introduce a parameter that measures the maximality of a code.

Definition

The **maximality degree** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ with $|\mathscr{C}| \geq 2$ is the integer defined by

$$\mu(\mathscr{C}) := \begin{cases} \min\{d(\mathscr{C}) - d(\mathscr{D}) \mid \mathscr{D} \subseteq \mathbb{F}_q^{n\times m} \text{ is a code with } \mathscr{D} \supsetneq \mathscr{C}\} & \text{if } \mathscr{C} \subsetneq \mathbb{F}_q^{n\times m}, \\ 1 & \text{if } \mathscr{C} = \mathbb{F}_q^{n\times m}. \end{cases}$$

Remarks:

- $\mu(\mathscr{C})$ is the "minimum price" (in terms of minimum distance) that one has to pay in order to enlarge $\mathscr{C}$ to a bigger code,
- $0 \leq \mu(\mathscr{C}) \leq d(\mathscr{C}) - 1$,
- $\mu(\mathscr{C}) > 0$ if and only if $\mathscr{C}$ is maximal.

Maximality

We introduce a parameter that measures the maximality of a code.

Definition

The **maximality degree** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ with $|\mathscr{C}| \geq 2$ is the integer defined by

$$\mu(\mathscr{C}) := \begin{cases} \min\{d(\mathscr{C}) - d(\mathscr{D}) \mid \mathscr{D} \subseteq \mathbb{F}_q^{n \times m} \text{ is a code with } \mathscr{D} \supsetneq \mathscr{C}\} & \text{if } \mathscr{C} \subsetneq \mathbb{F}_q^{n \times m}, \\ 1 & \text{if } \mathscr{C} = \mathbb{F}_q^{n \times m}. \end{cases}$$

Remarks:

- $\mu(\mathscr{C})$ is the "minimum price" (in terms of minimum distance) that one has to pay in order to enlarge $\mathscr{C}$ to a bigger code,
- $0 \leq \mu(\mathscr{C}) \leq d(\mathscr{C}) - 1$,
- $\mu(\mathscr{C}) > 0$ if and only if $\mathscr{C}$ is maximal.

Proposition (Byrne-R.)

For any code $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ with $|\mathscr{C}| \geq 2$ we have $\mu(\mathscr{C}) = d(\mathscr{C}) - \min\{\rho(\mathscr{C}),\ d(\mathscr{C})\}$.
In particular, if $\mathscr{C}$ is maximal then $\rho(\mathscr{C}) = d(\mathscr{C}) - \mu(\mathscr{C})$.

Translates of a code

For a code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$, let $W_i(\mathscr{C}) := |\{M \in \mathscr{C} \mid \mathrm{rk}(M) = i\}|$.

The **translate** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ by a matrix $X \in \mathbb{F}_q^{n\times m}$ is the code

$$\mathscr{C} + X := \{M + X : M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{n\times m}.$$

Remark

Full knowledge of the weight distribution of the translates of $\mathscr{C}$ tells us the covering radius, as

$$\rho(\mathscr{C}) = \max_{X\in\mathbb{F}_q^{n\times m}} \min_{N\in\mathscr{C}+X} \mathrm{rk}(N).$$

Even partial information may yield a bound on the covering radius.

Translates of a code

For a code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$, let $W_i(\mathscr{C}) := |\{M \in \mathscr{C} \mid \mathrm{rk}(M) = i\}|$.

The **translate** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ by a matrix $X \in \mathbb{F}_q^{n\times m}$ is the code

$$\mathscr{C} + X := \{M + X : M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{n\times m}.$$

Remark

Full knowledge of the weight distribution of the translates of $\mathscr{C}$ tells us the covering radius, as

$$\rho(\mathscr{C}) = \max_{X \in \mathbb{F}_q^{n\times m}} \min_{N \in \mathscr{C}+X} \mathrm{rk}(N).$$

Even partial information may yield a bound on the covering radius.

We now express the weight distribution

$$W_0(\mathscr{C}+X), \ldots, W_n(\mathscr{C}+X)$$

of the translate $\mathscr{C}+X$ of a linear code $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times n}$ in terms of

$$W_0(\mathscr{C}+X), \ldots, W_{n-d^\perp}(\mathscr{C}+X), \qquad \text{where } d^\perp = d(\mathscr{C}^\perp).$$

As an application, we obtain an upper bound on the covering radius of a linear code.

Translates of a code

Weight distribution of translates.

Theorem (Byrne-R.)

Let $\mathscr{C} \subsetneq \mathbb{F}_q^{n\times m}$ be a linear code, and let $X \in \mathbb{F}_q^{n\times m}$. Write $d^\perp := d(\mathscr{C}^\perp)$.
Then for all $i \in \{n-d^\perp+1, \dots, n\}$ we have

$$W_i(\mathscr{C}+X) = \sum_{u=0}^{n-d^\perp} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} n-u \\ i-u \end{bmatrix}_q \sum_{j=0}^{u} W_j(\mathscr{C}+X) \begin{bmatrix} u-j \\ n-j \end{bmatrix}_q + \\ + \sum_{u=n-d^\perp+1}^{i} \begin{bmatrix} n \\ u \end{bmatrix}_q \frac{|\mathscr{C}|}{q^{m(k-u)}}.$$

In particular, the distance distribution of the translate $\mathscr{C}+X$ is completely determined by n, m, $|\mathscr{C}|$ and the weights $W_0(\mathscr{C}+X), \dots, W_{n-d^\perp}(\mathscr{C}+X)$.

Translates of a code

Weight distribution of translates.

Theorem (Byrne-R.)

Let $\mathscr{C} \subsetneq \mathbb{F}_q^{n\times m}$ be a linear code, and let $X \in \mathbb{F}_q^{n\times m}$. Write $d^\perp := d(\mathscr{C}^\perp)$. Then for all $i \in \{n-d^\perp+1, \dots, n\}$ we have

$$W_i(\mathscr{C}+X) = \sum_{u=0}^{n-d^\perp} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} n-u \\ i-u \end{bmatrix}_q \sum_{j=0}^{u} W_j(\mathscr{C}+X) \begin{bmatrix} n-j \\ u-j \end{bmatrix}_q + \\ + \sum_{u=n-d^\perp+1}^{i} \begin{bmatrix} n \\ u \end{bmatrix}_q \frac{|\mathscr{C}|}{q^{m(k-u)}}.$$

In particular, the distance distribution of the translate $\mathscr{C}+X$ is completely determined by n, m, $|\mathscr{C}|$ and the weights $W_0(\mathscr{C}+X), \dots, W_{n-d^\perp}(\mathscr{C}+X)$.

Let $X \in \mathbb{F}_q^{n\times m} \notin \mathscr{C}$ be arbitrary. Then $W_0(\mathscr{C}+X) = 0$.

Apply the Theorem with $i := n-d^\perp+1$ and obtain:

Translates of a code and dual distance bound

For $X \in \mathbb{F}_q^{n\times m} \notin \mathscr{C}$ arbitrary:

$$W_{n+d^\perp+1}(\mathscr{C}+X) = \sum_{u=1}^{n-d^\perp} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} n-u \\ i-u \end{bmatrix}_q \sum_{j=1}^{u} W_j(\mathscr{C}+X) \begin{bmatrix} n-j \\ u-j \end{bmatrix}_q + \\ + \begin{bmatrix} n \\ n-d^\perp+1 \end{bmatrix}_q |\mathscr{C}|/q^{m(d^\perp-1)}.$$

Translates of a code and dual distance bound

For $X \in \mathbb{F}_q^{n\times m} \notin \mathscr{C}$ arbitrary:

$$W_{n+d^\perp+1}(\mathscr{C}+X) = \sum_{u=1}^{n-d^\perp} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} n-u \\ i-u \end{bmatrix}_q \sum_{j=1}^{u} W_j(\mathscr{C}+X) \begin{bmatrix} n-j \\ u-j \end{bmatrix}_q + \\ + \begin{bmatrix} n \\ n-d^\perp+1 \end{bmatrix}_q |\mathscr{C}|/q^{m(d^\perp-1)}.$$

In particular, $W_1(\mathscr{C}+X), \ldots, W_{n-d^\perp+1}(\mathscr{C}+X)$ cannot be all zero!

Since X was arbitrary, this implies the following.

Corollary (dual distance bound, Byrne-R.)

For any linear code $\mathscr{C} \subsetneq \mathbb{F}_q^{n\times m}$ we have $\rho(\mathscr{C}) \le n - d(\mathscr{C}^\perp) + 1$.

We have other bounds for linear / non-linear codes.

Initial sets

Let $a, b \in \mathbb{Z}_{>0}$ and $S \subseteq \{1, \ldots, a\} \times \{1, \ldots, b\}$. The **characteristic matrix** $\mathbb{I}(S) \in \mathbb{F}_2^{a \times b}$ of S is defined by

$$\mathbb{I}(S)_{ij} := \begin{cases} 1 & \text{if } (i,j) \in S, \\ 0 & \text{if } (i,j) \notin S \end{cases}$$

Initial sets

Let $a, b \in \mathbb{Z}_{>0}$ and $S \subseteq \{1, \dots, a\} \times \{1, \dots, b\}$. The **characteristic matrix** $\mathbb{I}(S) \in \mathbb{F}_2^{a \times b}$ of S is defined by

$$\mathbb{I}(S)_{ij} := \begin{cases} 1 & \text{if } (i,j) \in S, \\ 0 & \text{if } (i,j) \notin S \end{cases}$$

Moreover, we denote by $\lambda(S)$ the minimum number of lines (rows or columns) required to cover all the ones in $\mathbb{I}(S)$.

Example

Let $a = 2,\ b = 3$ and $S = \{(1,1),(1,2),(2,2),(2,3)\}$. Then

$$\mathbb{I}(S) := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in \mathbb{F}_2^{2 \times 3} \qquad \text{and} \qquad \lambda(S) = 2.$$

Initial sets

Let $a, b \in \mathbb{Z}_{>0}$ and $S \subseteq \{1, ..., a\} \times \{1, ..., b\}$. The **characteristic matrix** $\mathbb{I}(S) \in \mathbb{F}_2^{a \times b}$ of S is defined by

$$\mathbb{I}(S)_{ij} := \begin{cases} 1 & \text{if } (i,j) \in S, \\ 0 & \text{if } (i,j) \notin S \end{cases}$$

Moreover, we denote by $\lambda(S)$ the minimum number of lines (rows or columns) required to cover all the ones in $\mathbb{I}(S)$.

Example

Let $a = 2$, $b = 3$ and $S = \{(1,1),(1,2),(2,2),(2,3)\}$. Then

$$\mathbb{I}(S) := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in \mathbb{F}_2^{2 \times 3} \qquad \text{and} \qquad \lambda(S) = 2.$$

The **initial entry** of a matrix $M \in \mathbb{F}_q^{n \times m}$, $M \neq 0$, is

$$\text{in}(M) := \min\{(i,j) \in \{1, ..., n\} \times \{1, ..., m\} \mid M_{ij} \neq 0\} \qquad \text{lexicographically.}$$

Initial sets

Example

Let

$$M := \begin{bmatrix} 0 & 0 & 4 & 2 & 0 \\ 1 & 0 & 3 & 2 & 1 \end{bmatrix} \in \mathbb{F}_5^{2\times 5}$$

Then $\mathrm{in}(M) = (1, 3)$.

Initial sets

Example

Let

$$M := \begin{bmatrix} 0 & 0 & 4 & 2 & 0 \\ 1 & 0 & 3 & 2 & 1 \end{bmatrix} \in \mathbb{F}_5^{2\times 5}$$

Then $\mathrm{in}(M) = (1,3)$.

Definition

The **initial set** of a non-zero linear code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ is

$$\mathrm{in}(\mathscr{C}) := \{\mathrm{in}(M) \mid M \in \mathscr{C},\ M \neq 0\} \quad \subseteq \{1,\dots,n\} \times \{1,\dots,m\}.$$

Initial sets

Example

Let

$$M := \begin{bmatrix} 0 & 0 & 4 & 2 & 0 \\ 1 & 0 & 3 & 2 & 1 \end{bmatrix} \in \mathbb{F}_5^{2\times 5}$$

Then $\mathrm{in}(M) = (1,3)$.

Definition

The **initial set** of a non-zero linear code $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ is

$$\mathrm{in}(\mathscr{C}) := \{\mathrm{in}(M) \mid M \in \mathscr{C},\ M \neq 0\} \quad \subseteq \{1,\dots,n\} \times \{1,\dots,m\}.$$

First properties of the initial set.

Remark

Let $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ be a non-zero linear code. Then

$$\dim(\mathscr{C}) = |\mathrm{in}(\mathscr{C})|.$$

Initial set bound

Theorem (initial set bound, Byrne-R.)

Let $\{0\} \neq \mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ be a linear code. Let $S := \{1, \ldots, n - d(\mathscr{C}) + 1\} \times \{1, \ldots, m\} \setminus \mathrm{in}(\mathscr{C})$. Then

$$\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1 + \lambda(S).$$

Initial set bound

Theorem (initial set bound, Byrne-R.)

Let $\{0\} \neq \mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ be a linear code. Let $S := \{1,\dots,n-d(\mathscr{C})+1\} \times \{1,\dots,m\} \setminus \mathrm{in}(\mathscr{C})$. Then

$$\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1 + \lambda(S).$$

Example

Let $q=2$ and $n=m=3$. Let $\mathscr{C}$ be the linear code generated by

$$\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}, \quad \begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\0&1&1\\1&0&0\end{bmatrix}.$$

We have $d(\mathscr{C})=2$ and $\mathrm{in}(\mathscr{C}) = \{(1,1),(1,2),(2,1),(2,2)\}$.

Initial set bound

Theorem (initial set bound, Byrne-R.)

Let $\{0\} \neq \mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ be a linear code. Let $S := \{1, \ldots, n - d(\mathscr{C}) + 1\} \times \{1, \ldots, m\} \setminus \mathrm{in}(\mathscr{C})$. Then

$$\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1 + \lambda(S).$$

Example

Let $q = 2$ and $n = m = 3$. Let $\mathscr{C}$ be the linear code generated by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We have $d(\mathscr{C}) = 2$ and $\mathrm{in}(\mathscr{C}) = \{(1,1), (1,2), (2,1), (2,2)\}$. Therefore

$$S = \{1, \ldots, 2\} \times \{1, \ldots, 3\} \setminus \mathrm{in}(\mathscr{C}) = \{(1,3), (2,3)\}, \qquad \mathbb{I}(S) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Initial set bound

Theorem (initial set bound, Byrne-R.)

Let $\{0\} \neq \mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ be a linear code. Let $S := \{1,\dots,n-d(\mathscr{C})+1\} \times \{1,\dots,m\} \setminus \mathrm{in}(\mathscr{C})$. Then

$$\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1 + \lambda(S).$$

Example

Let $q=2$ and $n=m=3$. Let $\mathscr{C}$ be the linear code generated by

$$\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}, \quad \begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\0&1&1\\1&0&0\end{bmatrix}.$$

We have $d(\mathscr{C}) = 2$ and $\mathrm{in}(\mathscr{C}) = \{(1,1),(1,2),(2,1),(2,2)\}$. Therefore

$$S = \{1,\dots,2\} \times \{1,\dots,3\} \setminus \mathrm{in}(\mathscr{C}) = \{(1,3),(2,3)\}, \qquad \mathbb{I}(S) = \begin{bmatrix}0&0&1\\0&0&1\end{bmatrix}$$

So $\lambda(S) = 1$ and (by the Theorem) $\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1 + \lambda(S) = 2$.

The other bounds give $\rho(\mathscr{C}) \leq 3$.

Other results

If $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ is a linear code of dimension k and $m \gg 0$, then we can say what the "expected" covering radius of $\mathscr{C}$ is for $q \to +\infty$.

Other results

If $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ is a linear code of dimension k and $m \gg 0$, then we can say what the "expected" covering radius of $\mathscr{C}$ is for $q \to +\infty$.

Theorem (Byrne-R.)

Let $0 \leq k \leq nm$ be an integer. Denote by $\mathscr{F}$ the family of linear codes $\mathscr{C} \subseteq \mathbb{F}_q^{n \times m}$ of dimension k. Let $\mathscr{F}' := \{\mathscr{C} \in \mathscr{F} \mid \rho(\mathscr{C}) = n - \lfloor k/m \rfloor\}$. Then

$$\lim_{q \to +\infty} \frac{|\mathscr{F}'|}{|\mathscr{F}|} = 1 \qquad \text{whenever} \qquad k < (m - n + \lfloor k/m \rfloor + 1)(\lfloor k/m \rfloor + 1).$$

Other results

If $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ is a linear code of dimension k and $m \gg 0$, then we can say what the "expected" covering radius of $\mathscr{C}$ is for $q \to +\infty$.

Theorem (Byrne-R.)

Let $0 \leq k \leq nm$ be an integer. Denote by $\mathscr{F}$ the family of linear codes $\mathscr{C} \subseteq \mathbb{F}_q^{n\times m}$ of dimension k. Let $\mathscr{F}' := \{\mathscr{C} \in \mathscr{F} \mid \rho(\mathscr{C}) = n - \lfloor k/m \rfloor\}$. Then

$$\lim_{q\to+\infty} \frac{|\mathscr{F}'|}{|\mathscr{F}|} = 1 \qquad \text{whenever} \qquad k < (m-n+\lfloor k/m \rfloor+1)(\lfloor k/m \rfloor+1).$$

Thank you!