Network Coding, Rank-Metric Codes, and Rook Theory

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Outline

1. Network coding

2. Rank-metric codes and topics in combinatorics

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2. Rank-metric codes and topics in combinatorics

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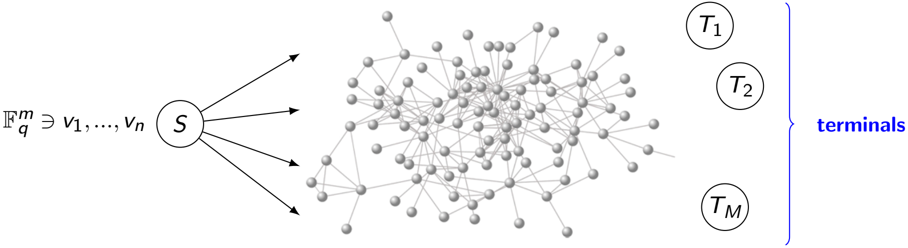

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- The terminals demand **all** the messages (multicast).

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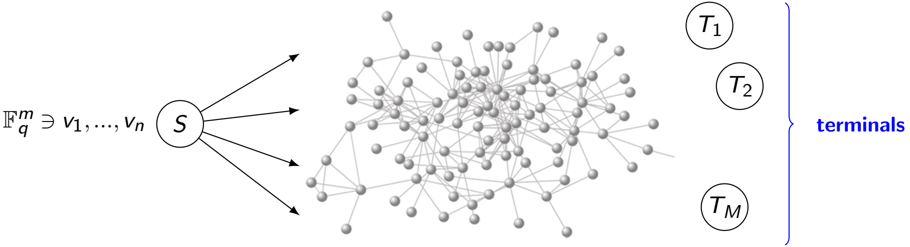

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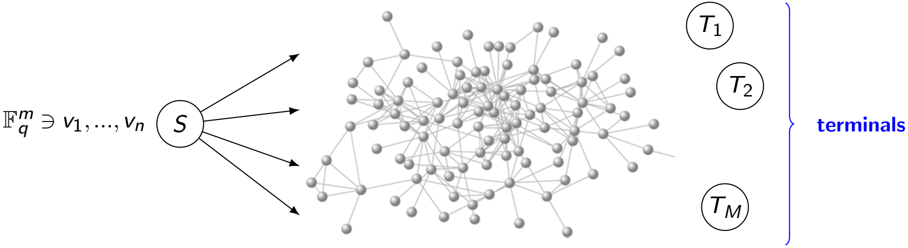

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Maximize the messages that are transmitted to **all** terminals per channel use (**rate**).

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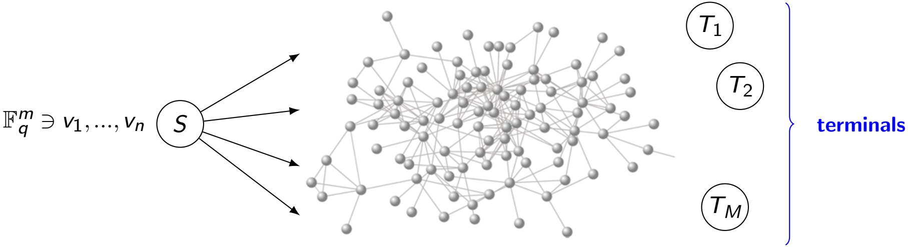

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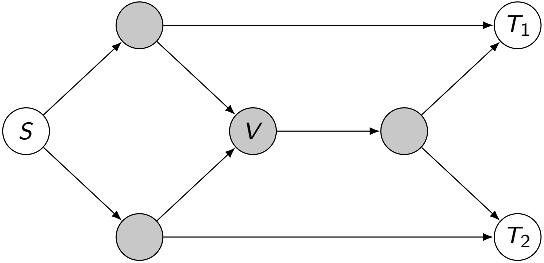
IDEA (Ahlswede-Cai-Li-Yeung 2000): allow the nodes to recombine packets.

The “Butterfly” network

v_1

v_2



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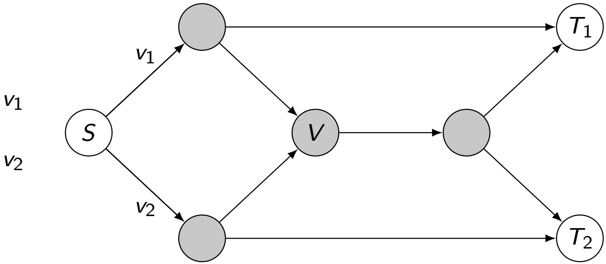

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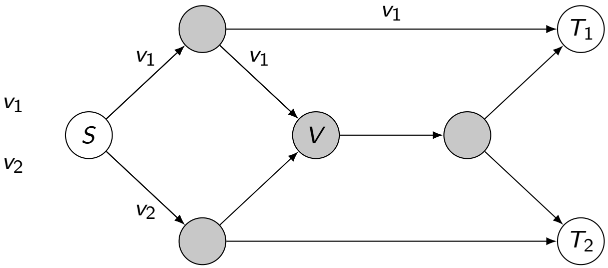

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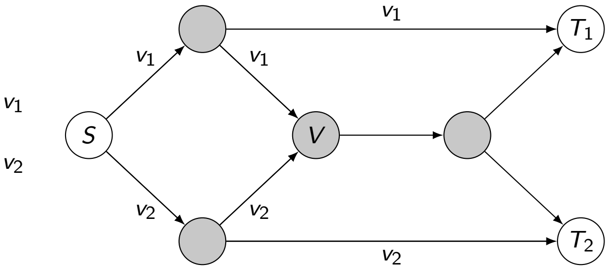

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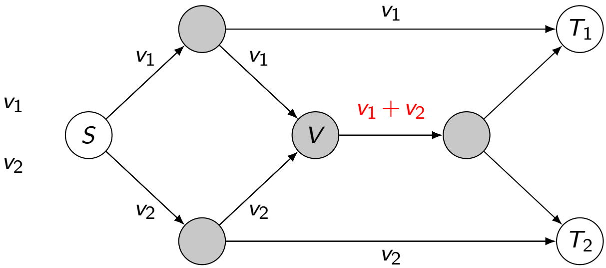

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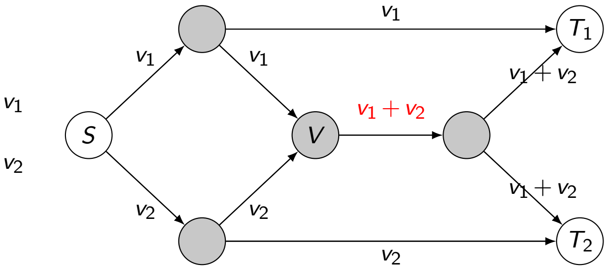

This strategy is better than routing.

Min-cut bound

- $\mathscr{N}$ the network
- S the source
- $\mathbf{T} = \{T_1, ..., T_M\}$ the set of terminals

Theorem (Ahlswede-Cai-Li-Yeung 2000)

The (multicast) rate of any communication over $\mathscr{N}$ satisfies

$$\text{rate} \le \mu(\mathscr{N}) := \min\{\text{min-cut}(S, T_i) \mid 1 \le i \le M\},$$

where min-cut(S, T_i) is the min. # of edges that one has to remove in $\mathscr{N}$ to disconnect S and T_i.

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YES, if $q \gg 0$. In fact, **linear operations** suffice.

Example

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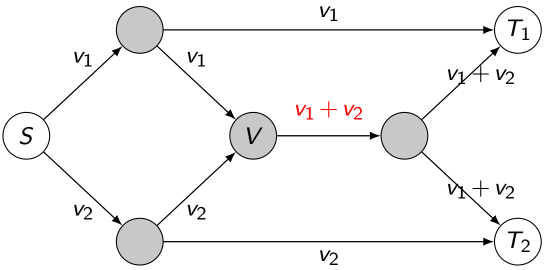

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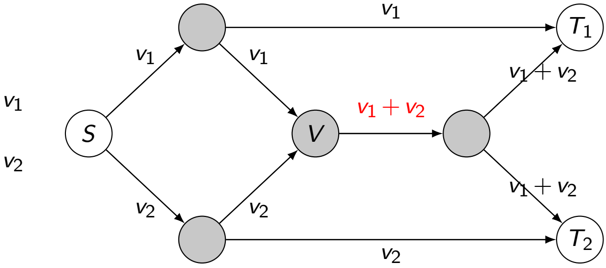

$$\text{min-cut}(S, T_1) = \text{min-cut}(S, T_2) = 2 \quad \Rightarrow \quad \mu(\mathcal{N}) = 2.$$

Therefore the strategy is optimal over any field $\mathbb{F}_q$.

Moreover, the node operations are linear.

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Then we can write:

$$\begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_{r(T)}^T \end{bmatrix} = G(T) \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

where $G(T) \in \mathbb{F}_q^{r(T) \times n}$ is the **transfer matrix** at T, describing all linear nodes operations.

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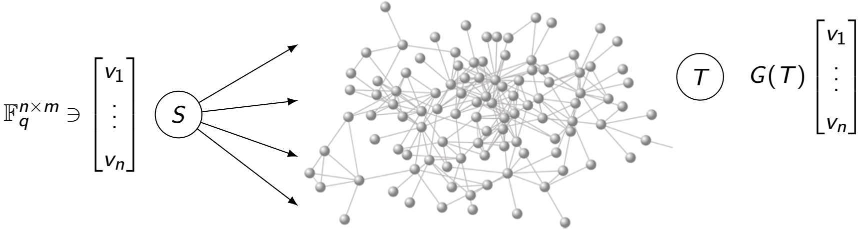
Theorem (Li-Yeung-Cai 2002; Kötter-Médard 2003)

1. Without loss of generality, $r(T) = n = \mu(\mathcal{N})$ for all $T \in \mathbf{T}$.
2. If $q \geq |\mathbf{T}|$, then there exist linear nodes operations such that $G(T)$ is a $n \times n$ invertible matrix for each terminal $T \in \mathbf{T}$, **simultaneously**.

Let $n = \mu(\mathcal{N})$.



where $G(T) \in \mathbb{F}_q^{n\times n}$ is invertible for every $T \in \mathbf{T} \quad (q \gg 0)$.

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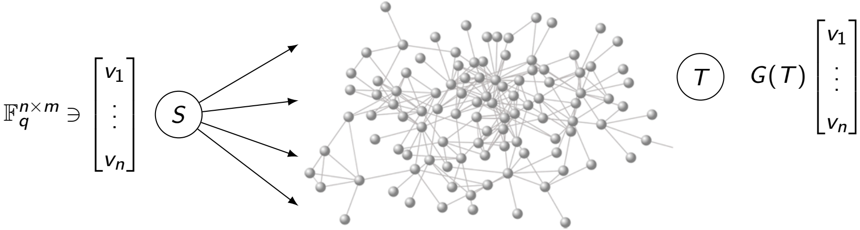

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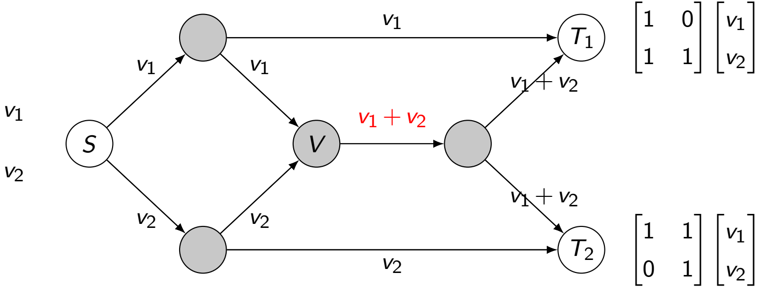
Decoding

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = G(T)^{-1} \left(G(T) \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right).$$

Each terminal $T \in \mathbf{T}$ computes the inverse of its own transfer matrix $G(T)$.

The max-flow-min-cut theorem



Error correction in networks

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The model

One adversary can change the value of up to t edges (t is the adversarial *strength*).

Other models are possible (restricted avdersaries, erasures, ...). We study these in:
Kschischang, R., *Adversarial Network Coding*, IEEE Trans. Inf. Th. 2018.

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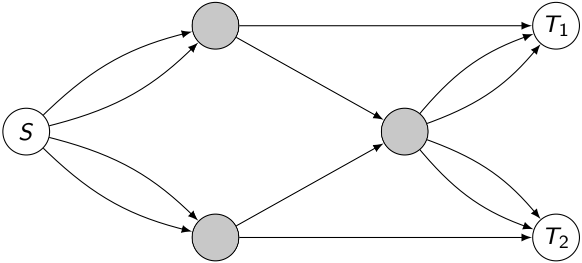

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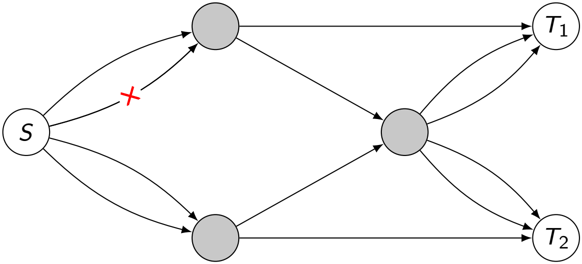

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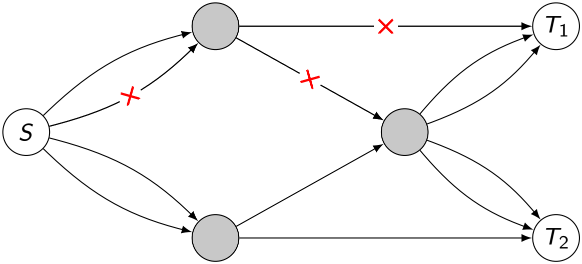

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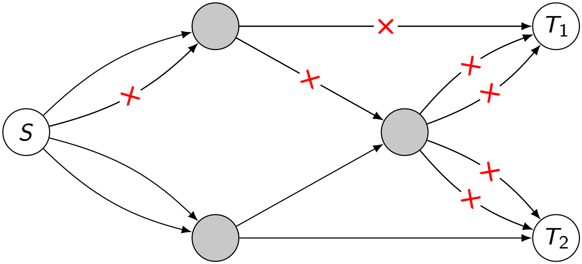

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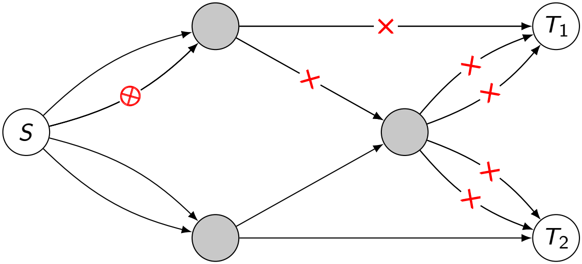

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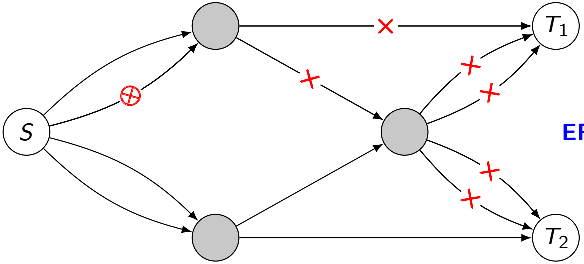

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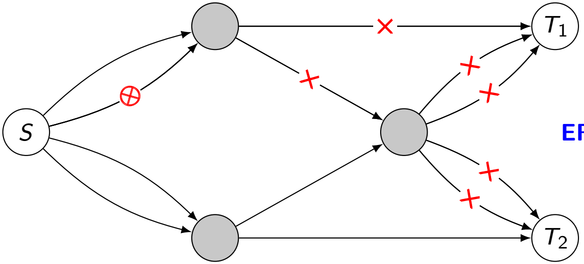

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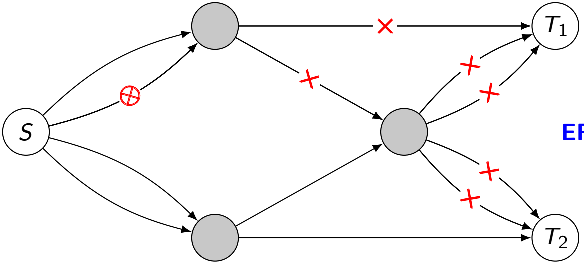
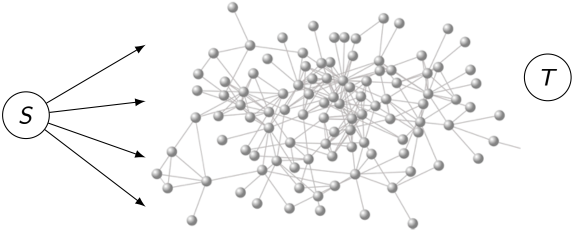


ERROR AMPLIFICATION

Natural solution: design the node operations carefully (decoding at intermediate nodes).
Other solution: use rank-metric codes.

Error correction in networks

Suppose we use linear network coding, $n = \mu(\mathcal{N})$.



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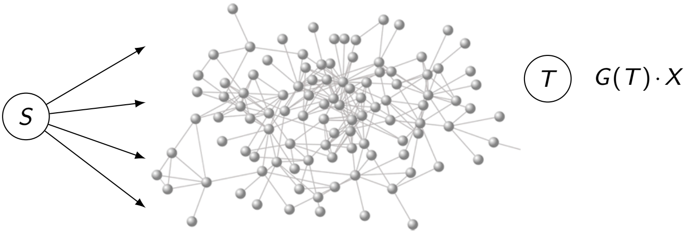

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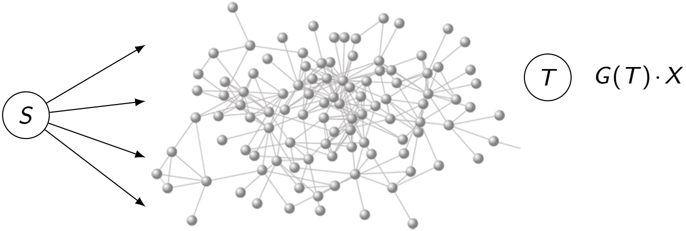

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In an error-free context: $\;X$ is sent, $\;G(T) \cdot X$ is received by terminal $T \in \mathbf{T}$.

If errors occur: $\;X$ is sent, $\;Y(T) \neq G(T) \cdot X$ is received by terminal $T \in \mathbf{T}$.

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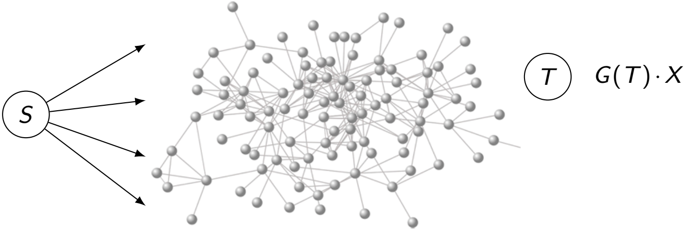

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Theorem (Silva-Kschischang-Koetter 2008)

If at most t edges were corrupted, then $\mathrm{rk}(Y(T) - G(T)\cdot X) \leq t$ for all $T \in \mathbf{T}$.

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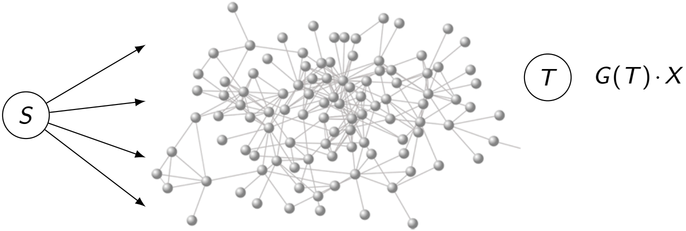

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IDEA: use the **rank metric** as a measure of the discrepancy between $Y(T)$ and $G(T) \cdot X$.

$$d_{\text{rk}}(A, B) = \text{rk}(A - B).$$

Definition

A **rank-metric code** is a non-zero $\mathbb{F}_q$-subspace $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$. Its **minimum distance** is

$$d_{\mathrm{rk}}(\mathscr{C}) = \min\{\mathrm{rk}(X) \mid X \in \mathscr{C},\, X \neq 0\}.$$

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Communication schemes based on rank-metric codes are:

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(2) compatible with linear network coding

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Theorem (R.-Kschischang)

For more general scenarios, there is no capacity-achieving scheme with (2) and (3).

E.g., multiple adversaries, erasure adversaries, or restricted adversaries.
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⚠ **ACHTUNG!** Noise is **adversarial**. Probabilistic models require different methods.

Remark

PROBABILISTIC INFORMATION THEORY	ZERO–ERROR INFORMATION THEORY
noise follows a probability distribution (e.g., binary symmetric channel)	noise is adversarial
allow small probability of decoding failure (message can be repeated)	probability does not make sense
satellites, phones, space missions, trains	adversaries (Byzantine attacks), storage
block codes with	the Hamming metric
not capacity–achieving	capacity–achieving
for networks: case–by–case theory	rank–metric codes

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$$d_{\mathrm{rk}}(\mathscr{C}) = \min\{\mathrm{rk}(M) \mid M \in \mathscr{C},\ M \neq 0\}.$$

Codes as math objects $\rightsquigarrow$ connections to other areas of mathematics:

- rank-metric codes and association schemes
- rank-metric codes and q-designs (also called subspace designs)
- rank-metric codes and lattices
- rank-metric codes and semifields
- rank-metric codes and q-rook polynomials
- rank-metric codes and q-polymatroids

(In the sequel, we assume $m \geq n$ w.l.o.g.)

Outline

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2. Rank-metric codes and topics in combinatorics

MacWilliams identities for the rank metric

Notion of duality in $\mathbb{F}_q^{n\times m}$: the **trace-product** of $M, N \in \mathbb{F}_q^{n\times m}$ is $\langle M, N\rangle := \mathrm{Tr}(MN^\top)$.

Definition

The **dual** of a rank-metric code $\mathscr{C} \le \mathbb{F}_q^{n\times m}$ is

$$\mathscr{C}^\perp := \{N \in \mathbb{F}_q^{n\times m} \mid \langle M, N\rangle = 0 \text{ for all } M \in \mathscr{C}\}.$$

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We count the number of rank i matrices in a rank-metric code:

$$W_i(\mathscr{C}) := |\{M \in \mathscr{C} \mid \mathrm{rk}(M) = i\}| \qquad \textbf{(rank enumerator)}$$

MacWilliams identities for the rank metric

Notion of duality in $\mathbb{F}_q^{n\times m}$: the **trace-product** of $M, N \in \mathbb{F}_q^{n\times m}$ is $\langle M, N\rangle := \mathrm{Tr}(MN^\top)$.

Definition

The **dual** of a rank-metric code $\mathscr{C} \leq \mathbb{F}_q^{n\times m}$ is

$$\mathscr{C}^\perp := \{N \in \mathbb{F}_q^{n\times m} \mid \langle M, N\rangle = 0 \text{ for all } M \in \mathscr{C}\}.$$

We count the number of rank i matrices in a rank-metric code:

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Theorem (Delsarte)

Let $\mathscr{C} \leq \mathbb{F}_q^{n\times m}$, and let $0 \leq j \leq n$. we have

$$W_j(\mathscr{C}^\perp) \;=\; \frac{1}{|\mathscr{C}|} \sum_{i=0}^{n} W_i(\mathscr{C}) \sum_{s=0}^{n} (-1)^{j-s}\; q^{ms+\binom{j-s}{2}} \begin{bmatrix} n-i \\ s \end{bmatrix}_q \begin{bmatrix} n-s \\ j-s \end{bmatrix}_q .$$

Original proof by Delsarte uses association schemes and recurrence relations.

MacWilliams identities for the rank metric

For a code $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$ and a subspace $U \leq \mathbb{F}_q^n$, let

$$\begin{aligned} f_{\mathscr{C}}(U) &:= |\{M \in \mathscr{C} \mid \text{col-space}(M) = U\}| \\ g_{\mathscr{C}}(U) &:= \sum_{V \leq U} f_{\mathscr{C}}(V) = |\{M \in \mathscr{C} \mid \text{col-space}(M) \subseteq U\}| \end{aligned}$$

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where μ is the Mœbius function of the lattice of subspaces of $\mathbb{F}_q^n$.

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Proposition (R.)

$$g_{\mathscr{C}^\perp}(V) = \frac{q^{m \cdot \dim(V)}}{|\mathscr{C}|}\, g_{\mathscr{C}}(V^\perp),$$

where $V^\perp$ is the orthogonal of $V \leq \mathbb{F}_q^n$ w. r. to the standard inner product of $\mathbb{F}_q^n$.

MacWilliams identities for the rank metric

$$W_j(\mathscr{C}^\perp) = \frac{1}{|\mathscr{C}|} \sum_{i=0}^{j} (-1)^{j-i} q^{mi+\binom{j-i}{2}} \sum_{\substack{U \leq \mathbb{F}_q^n \\ \dim(U)=j}} \sum_{\substack{V \leq U \\ \dim(V)=i}} g_{\mathscr{C}}(V^\perp)$$

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Why a new proof?

E.g., the technique can be “exported” to other contexts (**pivot enumerators**).

But before looking at other types of MacWilliams identities...

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Examples of problems

Compute the number of rank r matrices $M \in \mathbb{F}_q^{n \times m}$ such that:

- their entries sum to zero, or
- a certain set of diagonal entries are zero ($M_{ii} = 0$ for all $i \in I \subseteq \{1, \dots, n\}$), or
- ...

Theorem (R.)

Let $\emptyset \neq I \subseteq \{1,\dots,n\}$. The number of rank r matrices $M \in \mathbb{F}_q^{n \times m}$ with $M_{ii} = 0$ for all $i \in I$ is given by the formula

$$v_r(I) := q^{-|I|} \sum_{i=0}^{|I|} \binom{|I|}{i} (q-1)^i \sum_{s=0}^{n} (-1)^{r-s} q^{ms+\binom{r-s}{2}} \begin{bmatrix} n-s \\ n-r \end{bmatrix}_q \begin{bmatrix} n-i \\ s \end{bmatrix}_q .$$

MacWilliams identities for the rank metric

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Let $\mathscr{C}[I]$ be the space of matrices supported on $\{(i,i) \mid i \in I\}$.

Then $\mathscr{C}[I] \leq \mathbb{F}_q^{n\times m}$ is a linear rank-metric code, and

$$v_r(I) = W_r(\mathscr{C}[I]^\perp)$$

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Now, $|\mathscr{C}[I]| = q^{|I|}$ and $W_i(\mathscr{C}[I]) = \binom{|I|}{i}(q-1)^i$ for all i.

MacWilliams identities for the rank metric

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Now, $\quad |\mathscr{C}[I]| = q^{|I|} \quad$ and $\quad W_i(\mathscr{C}[I]) = \binom{|I|}{i}(q-1)^i$ for all i.

The argument was further extended by Lewis and Morales (2017).

MacWilliams-type identities

MacWilliams-type identities have been extensively studied in the coding theory literature in various contexts:

- additive codes in finite abelian groups (discrete Fourier analysis),
- association schemes (Bose-Mesner algebras),
- regular lattices (support maps),
- posets (metric spaces from orders),
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Ingredients:

- a structured ambient space A
- a dual ambient space $\widehat{A}$
- a notion of duality: $\mathscr{C} \subseteq A$ yields $\mathscr{C}^\perp \subseteq \widehat{A}$
- counting devices on A and $\widehat{A}$ (e.g., the rank enumerator)

The pivot partition

For us, $A = \widehat{A} = \mathbb{F}_q^{n \times m}$. Duality is again trace-duality: $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$ yields $\mathscr{C}^\perp \leq \mathbb{F}_q^{n \times m}$.

We partition the elements of $\mathbb{F}_q^{n \times m}$ according to the pivot indices in their reduced row-echelon form. This defines a partition $\mathscr{P}^{\text{piv}}$ on $\mathbb{F}_q^{n \times m}$. Note:

$$|\mathscr{P}^{\text{piv}}| = \sum_{r=0}^{n} \binom{m}{r}.$$

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Example:

$$M = \begin{pmatrix} 1 & \bullet & 0 & 0 & \bullet \\ 0 & 0 & 1 & 0 & \bullet \\ 0 & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{piv}(M) = (1, 3, 4).$$

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Notation

$\Pi = \{(j_1, \dots, j_r) \mid 1 \leq r \leq n, \ 1 \leq j_1 < j_2 < \cdots < j_r \leq m\} \cup \{()\}$. Then $\mathscr{P}^{\text{piv}} = (P_\lambda)_{\lambda \in \Pi}$.

For a code $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$ and $\lambda \in \Pi$, $\mathscr{P}^{\text{piv}}(\mathscr{C}, \lambda) := |\mathscr{C} \cap P_\lambda|$.

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$\mathscr{P}^{\text{rpiv}}$ partitions the elements of $\mathbb{F}_q^{n\times m}$ according to the pivot indices in their reduced row-echelon form **computed from the right**.

$$\mathscr{P}^{\text{rpiv}} = (Q_\mu)_{\mu\in\Pi}, \qquad \mathscr{P}^{\text{rpiv}}(\mathscr{C},\mu) := |\mathscr{C}\cap Q_\mu|.$$

Theorem (Gluesing-Luerssen, R.)

Let $\mathscr{C} \leq \mathbb{F}_q^{n\times m}$, and let $\lambda,\mu \in \Pi$. We have

$$\mathscr{P}^{\text{rpiv}}(\mathscr{C}^\perp,\mu) = \frac{1}{|\mathscr{C}|}\sum_{\lambda\in\Pi} K(\lambda,\mu)\cdot\mathscr{P}^{\text{piv}}(\mathscr{C},\lambda)$$

for suitable integers $K(\lambda,\mu)$. Moreover

$$(K(\lambda,\mu))_{\lambda,\mu}$$

is an invertible square matrix.

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Computing $K(\lambda,\mu)$...

The pivot partition

Definition

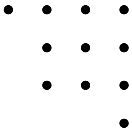
A **Ferrers diagram** is a subset $\mathscr{F} \subseteq [n] \times [m]$ that satisfies the following:

1. if $(i,j) \in \mathscr{F}$ and $j < m$, then $(i,j+1) \in \mathscr{F}$ (right aligned),
2. if $(i,j) \in \mathscr{F}$ and $i > 1$, then $(i-1,j) \in \mathscr{F}$ (top aligned).

We represent a Ferrers diagram by its column lengths, $\mathscr{F} = [c_1, \ldots, c_m]$.

E.g.

$\mathscr{F} =$



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We denote by $\mathbb{F}_q[\mathscr{F}]$ the space of matrices supported on $\mathscr{F}$, and let

$$P_r(\mathscr{F};q) := \{M \in \mathbb{F}_q[\mathscr{F}] \mid \mathrm{rk}(M) = r\}.$$

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We can express $K(\lambda,\mu)$ in terms of $P_r(\mathscr{F};q)$, for certain r and for a suitable diagram $\mathscr{F}$.

The pivot partition

Theorem (Gluesing-Luerssen, R.)

Let $\lambda, \mu \in \Pi$. Set

$$\sigma = [m] \setminus \mu, \qquad \lambda \cap \sigma = (\lambda_{\alpha_1}, \dots, \lambda_{\alpha_x}), \qquad \mu \setminus \lambda = (\mu_{\beta_1}, \dots, \mu_{\beta_y}).$$

Furthermore, set

$$z_j = |\{i \in [x] \mid \lambda_{\alpha_i} < \mu_{\beta_j}\}| \;\text{ for } j \in [y], \qquad \mathscr{F} = [z_1, \dots, z_y].$$

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Then

$$K(\lambda, \mu) = \sum_{t=0}^{m} (-1)^{|\lambda|-t} q^{nt+\binom{|\lambda|-t}{2}} \sum_{r=0}^{|\lambda\cap\sigma|} P_r(\mathscr{F}; q) \begin{bmatrix} |\lambda \cap \sigma| - r \\ t \end{bmatrix}_q .$$

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$P_r(\mathscr{F}; q) \;\rightarrow\;$ **rook theory**

Definition

The ***q*-rook polynomial** associated with $\mathscr{F}$ and $r \geq 0$ is

$$R_r(\mathscr{F}) = \sum_{C \in \mathrm{NAR}_r(\mathscr{F})} q^{\mathrm{inv}(C,\mathscr{F})} \in \mathbb{Z}[q],$$

where:

- $\mathrm{NAR}_r(\mathscr{F})$ is the set of all placements of r non-attacking rooks on $\mathscr{F}$ (non-attacking means that no two rooks are in the same column, and no two are in the same row)
- $\mathrm{inv}(C, \mathscr{F}) \in \mathbb{N}$ is computed as shown on the board

q-Rook Polynomials

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Theorem (Haglund)

For any Ferrers diagram $\mathscr{F}$ and any $r \geq 0$ we have

$$P_r(\mathscr{F};q) = (q-1)^r \, q^{|\mathscr{F}|-r} \, R_r(\mathscr{F};q)_{|q^{-1}}$$

in the ring $\mathbb{Z}[q,q^{-1}]$.

Natural task: find an explicit expression for $R_r(\mathscr{F};q)$.

An explicit formula for $R_r(\mathscr{F})$:

Theorem (Gluesing-Luerssen, R.)

Let $\mathscr{F} = [c_1, \ldots, c_m]$ be an $n \times m$-Ferrers diagram. For $k \in [m]$ define $a_k = c_k - k + 1$.

For $j \in [m]$ let $\sigma_j \in \mathbb{Q}[x_1, \ldots, x_m]$ be the j^{th} elementary symmetric polynomial in m indeterminates ($\sigma_0 = 1$, ..., $\sigma_m = x_1 \cdots x_m$).

Then

$$R_r(\mathscr{F}; q) = \frac{q^{\binom{r+1}{2} - rm + \text{area}(\mathscr{F})}(-1)^{m-r}}{(1-q)^r \prod_{k=1}^{m-r}(1-q^k)} \sum_{t=m-r}^{m} (-1)^t \sigma_{m-t}(q^{-a_1}, \ldots, q^{-a_m}) \prod_{j=0}^{m-r-1}(1-q^{t-j}).$$

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Combining this with Haglund's theorem we find an explicit expression for $P_r(\mathscr{F}; q)$.

Proof is technical.

q-Rook Polynomials

A different approach: compute $P_r(\mathscr{F};q)$ directly. Notation: $\mathscr{F}=[c_1,\dots,c_m]$.

Theorem (Gluesing-Luerssen, R.)

$$P_r(\mathscr{F};q)=\sum_{1\le i_1<\cdots<i_r\le m} q^{rm-\sum_{j=1}^r i_j} \prod_{j=1}^{r}(q^{c_{i_j}-j+1}-1).$$

Proof is short.

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But inverting Haglund's theorem we also find a simple explicit formula for $R_r(\mathscr{F};q)$!

Corollary (Gluesing-Luerssen, R.)

$$R_r(\mathscr{F};q) = \frac{q^{\sum_{j=1}^m c_j - rm} \displaystyle\sum_{1 \le i_1 < \cdots < i_r \le m} \prod_{j=1}^{r} (q^{i_j + j - c_{i_j} - 1} - q^{i_j})}{(1-q)^r}.$$

q-Stirling Numbers

We can use these results to derive an explicit formula for the q-Stirling numbers of the second kind. The latter are defined via the recursion

$$S_{m+1,r} = q^{r-1} S_{m,r-1} + \frac{q^r - 1}{q - 1} S_{m,r}$$

with initial conditions $S_{0,0}(q) = 1$ and $S_{m,r}(q) = 0$ for $r < 0$ or $r > m$.

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Theorem (Garsia, Remmel)

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Thank you very much!