Network Coding and the Combinatorics of Codes

Alberto Ravagnani

UCD Algebra and Number Theory Seminar, **Oct. 2019**

What is coding theory?

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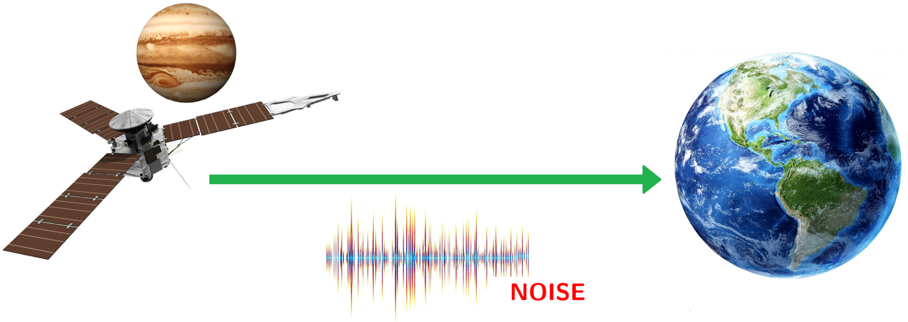

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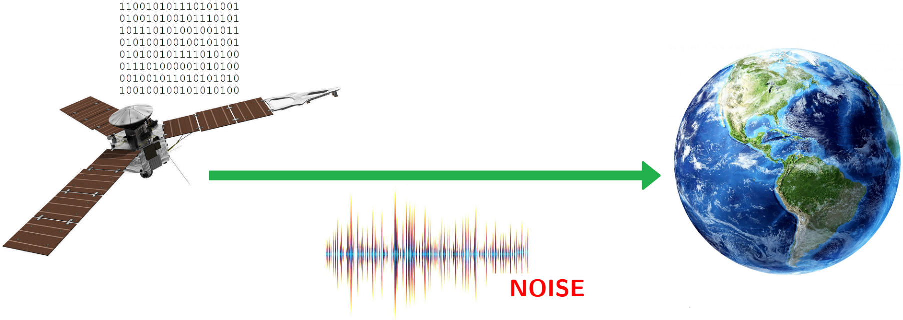

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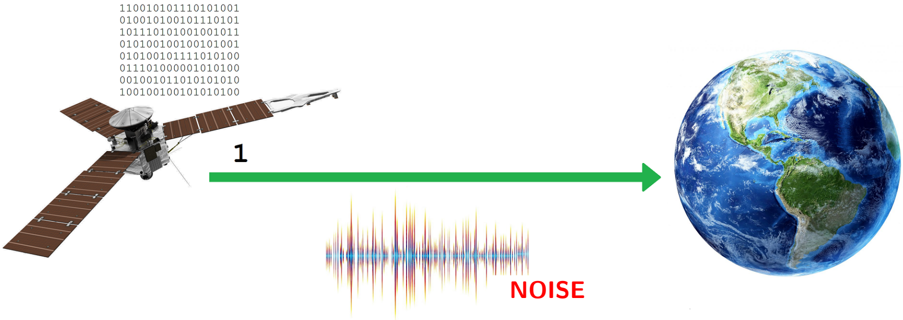

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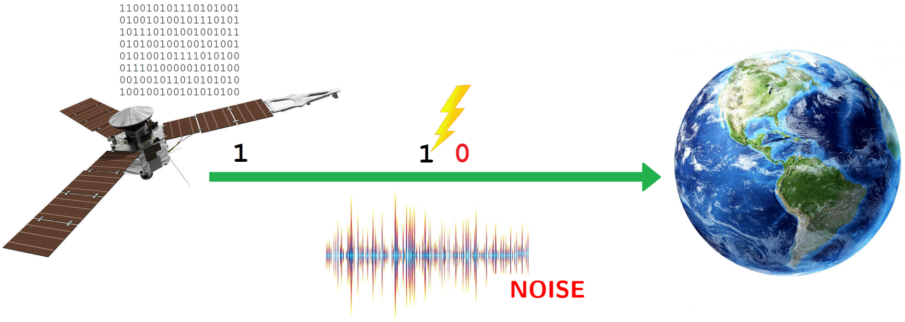

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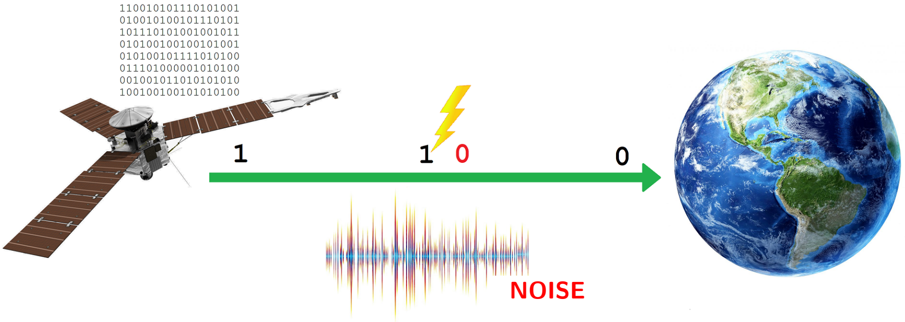

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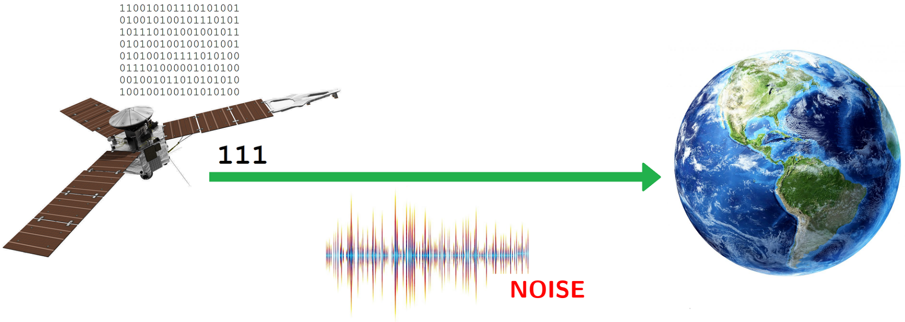

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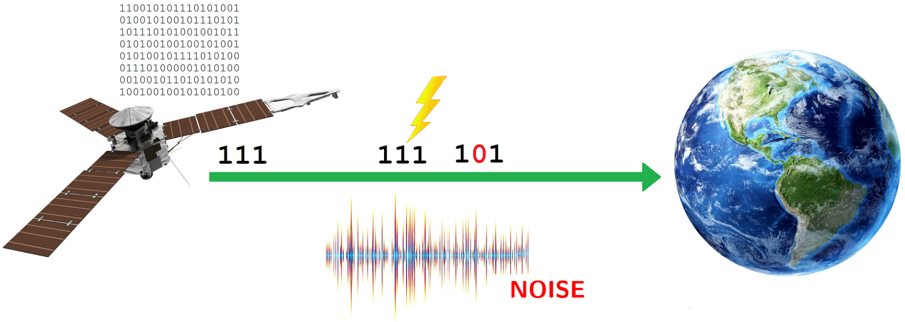

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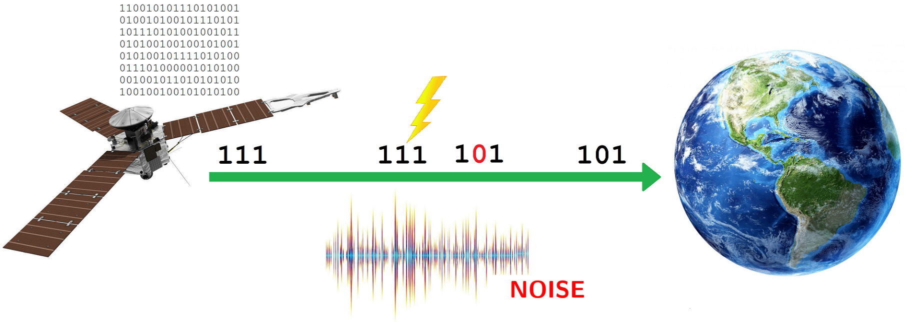

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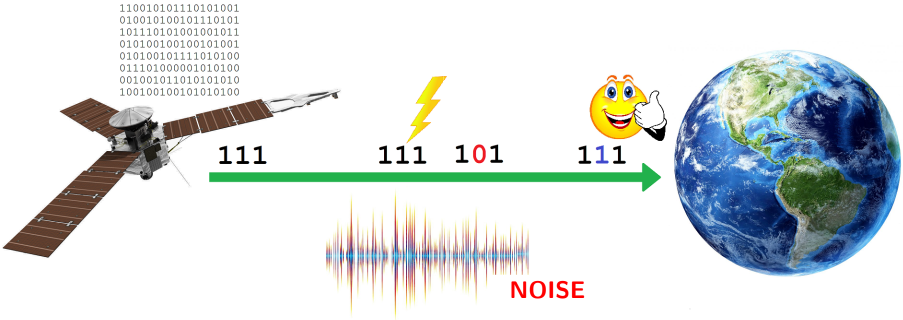

Error-correcting codes

Idea behind coding theory: add **redundancy**.

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Example: binary 3-time repetition scheme

$E : \mathbb{F}_2 \to \mathbb{F}_2^3, \quad E(a) = (a, a, a)$ for all $a \in \mathbb{F}_2$.

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Note: the image of E is a k-dimensional subspace of $\mathbb{F}_q^n$.

Example [continued]

$E(\mathbb{F}_2) = \{(0,0,0),(1,1,1)\}$.

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Definition

A **code** is an $\mathbb{F}_q$-linear subspace $\mathscr{C} \leq \mathbb{F}_q^n$. Elements of $\mathscr{C}$: **codewords**.

(we often forget about E)

Error-correcting codes

In a good quality code $\mathscr{C} \leq \mathbb{F}_q^n$, vectors are "far apart"...

Definition

- The **Hamming distance** between vectors $x, y \in \mathbb{F}_q^n$ is $d_{\mathrm{H}}(x,y) = \#\{i \mid x_i \neq y_i\}$.

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- The **minimum Hamming distance** of a code $\mathscr{C} \neq \{0\}$ is the integer

$$d_{\mathrm{H}}(\mathscr{C}) \;=\; \min\{d_{\mathrm{H}}(x,y) \mid x, y \in \mathscr{C}\ x \neq y\} \;=\; \min\{\omega_{\mathrm{H}}(x) \mid x \in \mathscr{C},\ x \neq 0\}.$$

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Note: a code $\mathscr{C}$ corrects up to $\lfloor (d-1)/2 \rfloor$ errors, where $d = d_{\mathrm{H}}(\mathscr{C})$.

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Note: a code $\mathscr{C}$ corrects up to $\lfloor (d-1)/2 \rfloor$ errors, where $d = d_{\mathrm{H}}(\mathscr{C})$.

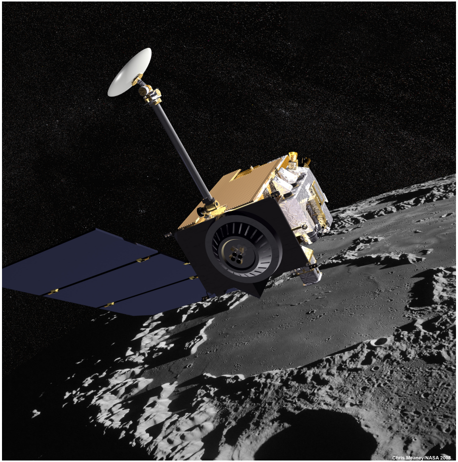
Theorem (Singleton, Komamiya)

Let $\mathscr{C} \leq \mathbb{F}_q^n$ be a non-zero code. Then $\dim(\mathscr{C}) \leq n - d_{\mathrm{H}}(\mathscr{C}) + 1$.

If $\mathscr{C}$ meets the bound with equality, then it is called an **MDS** code.

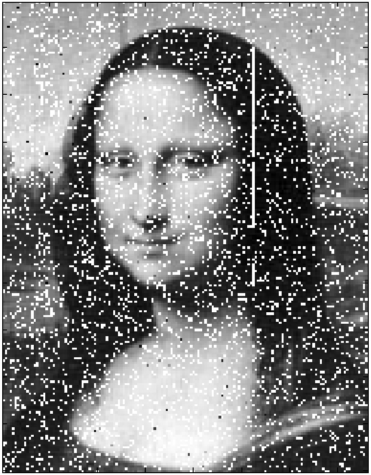
A concrete example

The LRO (Lunar Reconnaissance Orbiter) is taking pictures of the Moon...



A concrete example

Test of quality of transmissions:



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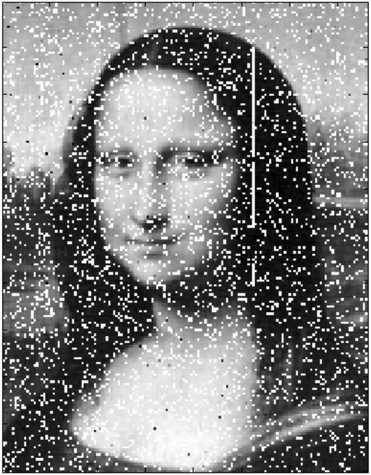

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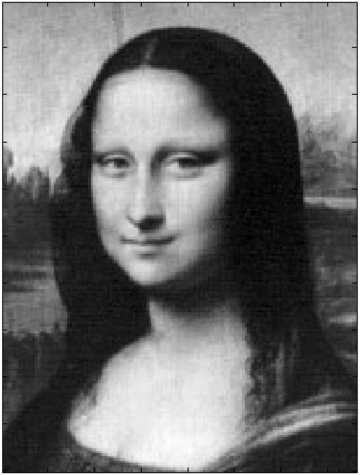

with coding

Network communication

Classical coding theory: **one** source of information, **one** terminal.

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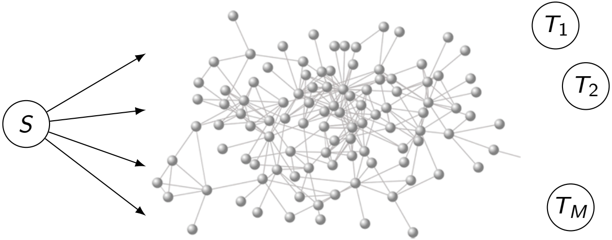
Network coding: **one/multiple** sources of information, **multiple** terminals.



Applications: LTE (mobile phones), distributed storage, peer-to-peer, streaming,...

Network coding

Network coding: data transmission over (noisy/lossy) networks

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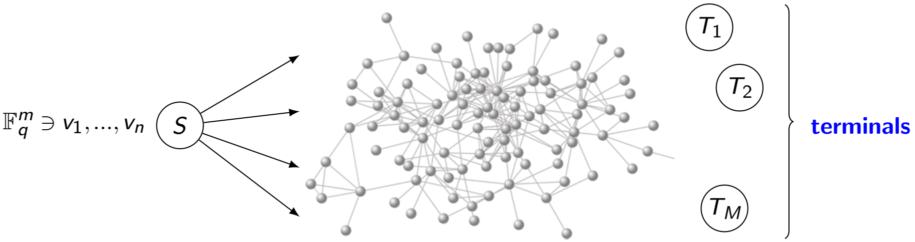
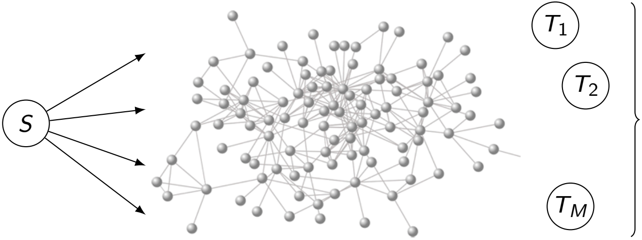


- One source S attempts to transmit messages $v_1, \ldots, v_n \in \mathbb{F}_q^m$.
- The terminals demand **all** the messages (multicast).

Network coding: data transmission over (noisy/lossy) networks

$\mathbb{F}_q^m \ni v_1, \dots, v_n$



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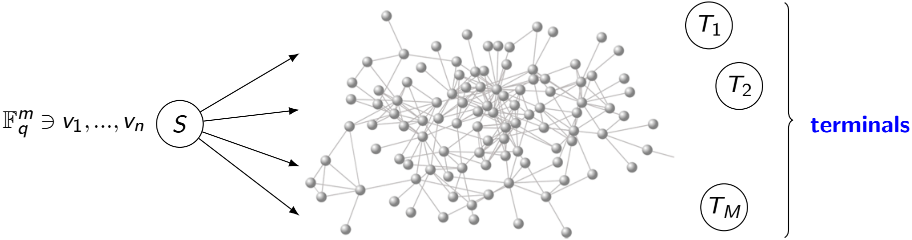

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Maximize the number of transmitted messages per channel use (**rate**).

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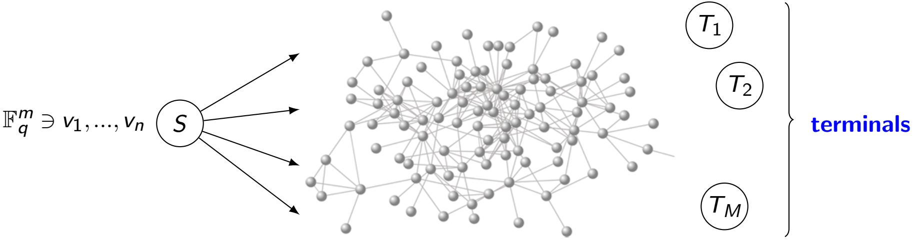

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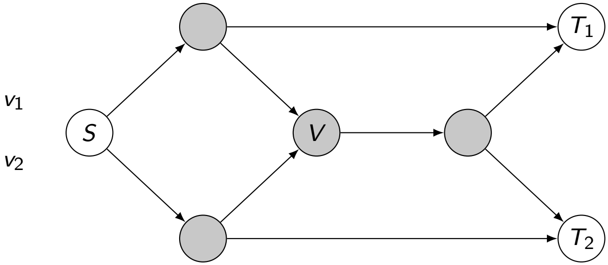
IDEA (Ahlswede-Cai-Li-Yeung 2000): allow the nodes to recombine packets.

The "Butterfly" network



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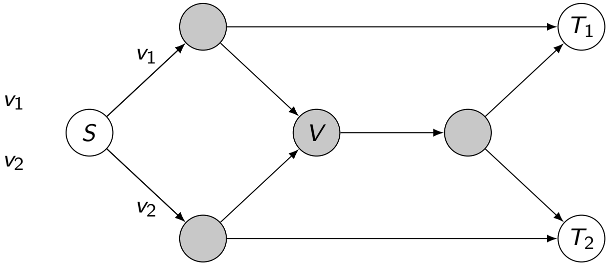

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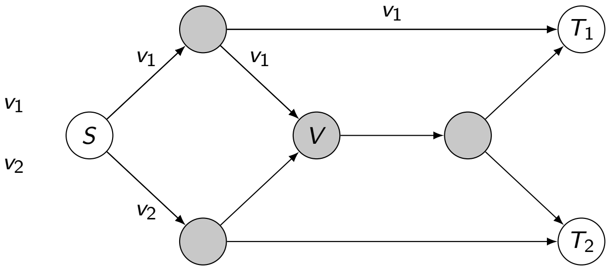

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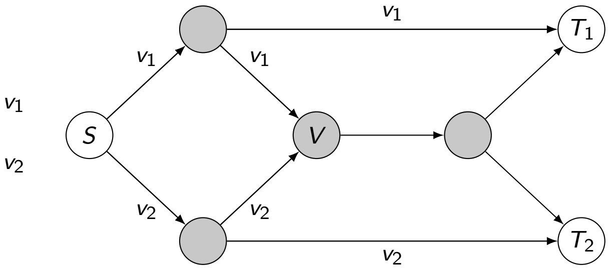

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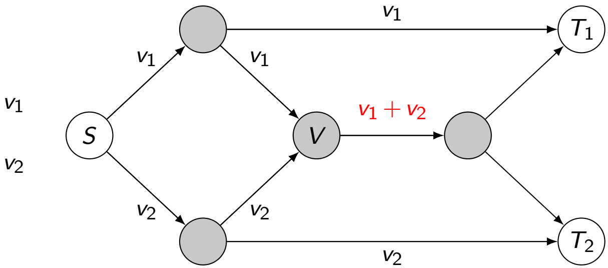

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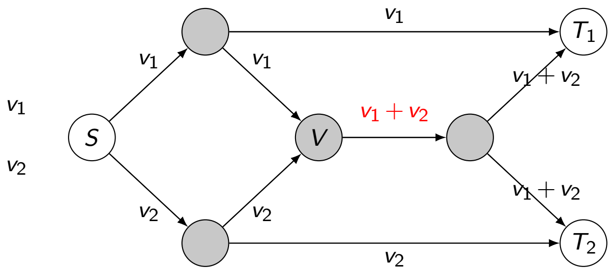

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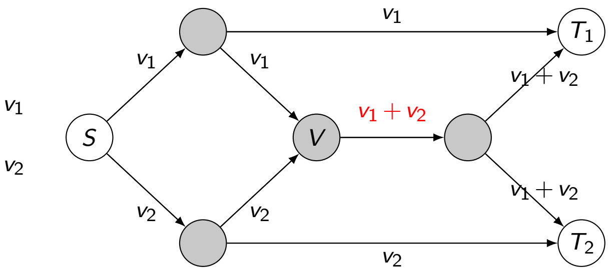

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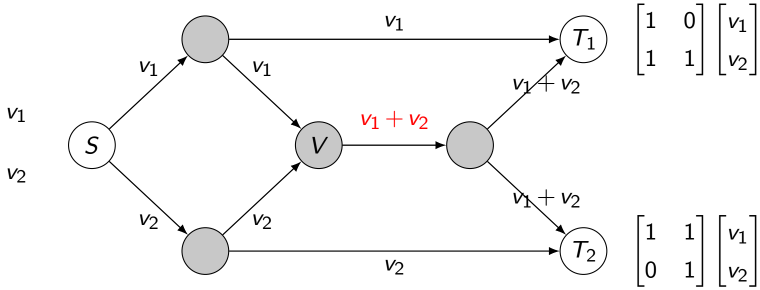

Note: This strategy is better than routing.

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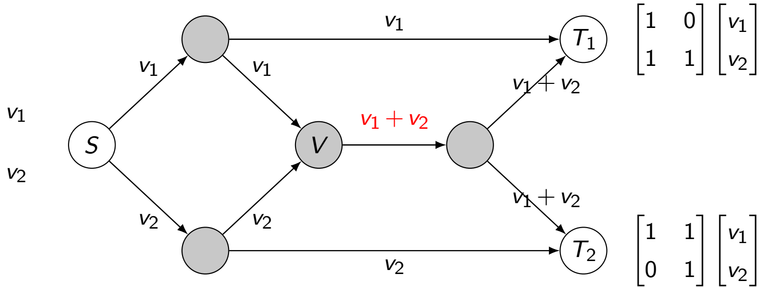

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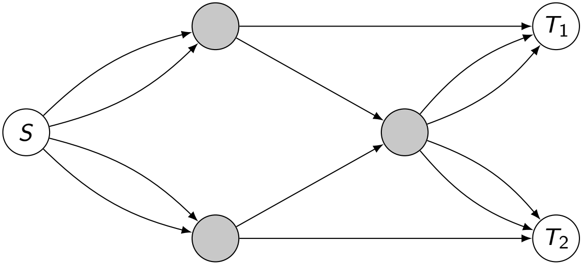
Theorem (Li-Yeung-Cai 2002, Koetter-Médard 2003)

This strategy (**linear** network coding) applies to general networks and is capacity achieving (w.r. to certain models), provided that $q \gg 0$.

Also, efficient algorithms to design the network operations are known.

Error correction in networks



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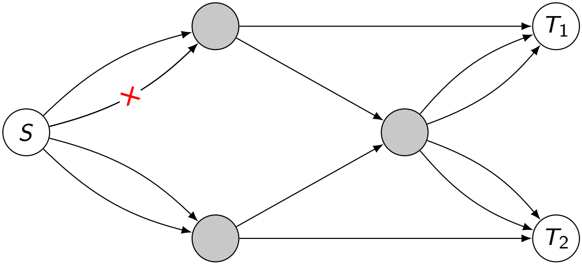

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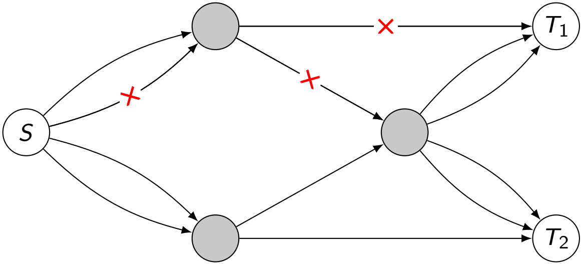

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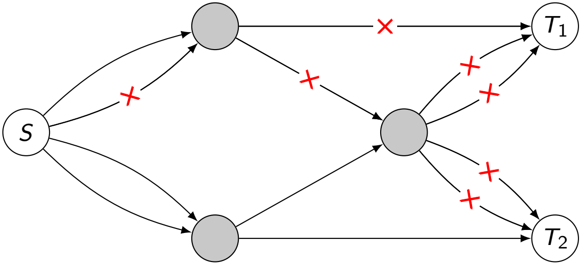

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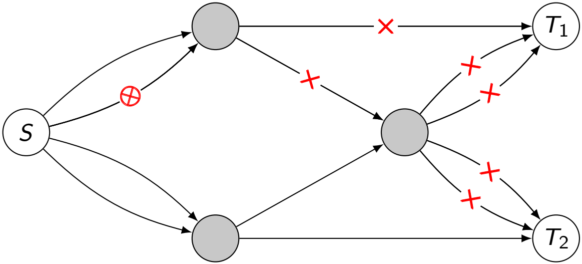

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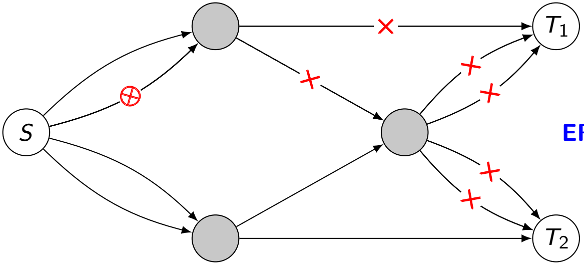
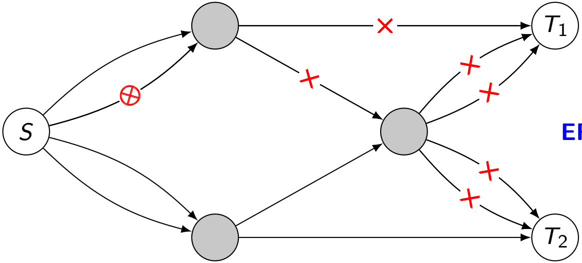


ERROR AMPLIFICATION

Error correction in networks



ERROR AMPLIFICATION

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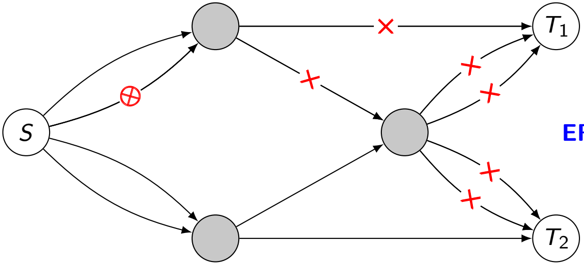

ERROR AMPLIFICATION

Natural solution: design the node operations carefully (decoding at intermediate nodes).

Other solution: use rank-metric codes.

Rank-metric codes

Definition

A **rank-metric code** is an $\mathbb{F}_q$-subspace $\mathscr{C} \leq \mathbb{F}_q^{n\times m}$. If $\mathscr{C} \neq \{0\}$, then its **minimum rank distance** is

$$d_{\mathrm{rk}}(\mathscr{C}) = \min\{\mathrm{rk}(X) \mid X \in \mathscr{C}, X \neq 0\}.$$

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Remark 1: for some scenarios, there is no communication scheme based on rk-metric codes with both (1) and (2). E.g., geographically restricted errors, erasures, ...

Kschischang, R., *Adversarial Network Coding*, IEEE Trans. Inf. Th. 2018.

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Remark 2: wireless networks are a very different story

Gorla, R., *An Algebraic Framework for End-to-End PLNC*, IEEE Trans. Inf. Th. 2018.

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NOTATION: $2 \leq n \leq m$ integers.

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There is a rank-analogue of the Singleton bound:

Theorem (Delsarte)

Let $\mathscr{C} \leq \mathbb{F}_q^{n\times m}$ be a non-zero rank-metric code. We have

$$\dim(\mathscr{C}) \leq m(n - d_{\mathrm{rk}}(\mathscr{C}) + 1).$$

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A code $\mathscr{C}$ is **MRD** if it meets the bound with equality $\quad (\Longrightarrow \dim(\mathscr{C}) \equiv 0 \mod m)$.

Classes of codes

Hamming space

- $\mathbb{F}_q^n, \quad d_{\mathrm{H}}(x,y) = |\{i \mid x_i \neq y_i\}|$
- Code: $\mathbb{F}_q$-subspace $\mathscr{C} \leq \mathbb{F}_q^n$
- Bound: $\dim(\mathscr{C}) \leq n - d_{\mathrm{H}}(\mathscr{C}) + 1$
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Classes of codes

Hamming space

- $\mathbb{F}_q^n, \quad d_{\mathrm{H}}(x,y) = |\{i \mid x_i \neq y_i\}|$
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Vector rank-metric space

- $\mathbb{F}_{q^m}^n$ with $m \geq n, \quad d_{\mathrm{rk}}(x,y) = \dim_{\mathbb{F}_q} \mathrm{span}\{x_1 - y_1, \dots, x_n - y_n\}$
- Code: $\mathbb{F}_{q^m}$-subspace $\mathscr{C} \leq \mathbb{F}_{q^m}^n$
- Bound: $\dim_{\mathbb{F}_{q^m}}(\mathscr{C}) \leq n - d_{\mathrm{rk}}(\mathscr{C}) + 1$
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Density of MDS codes

A randomly chosen k-dimensional code is MDS with high probability, if $q \gg 0$.

Theorem (Folklore)

Let $n \geq k \geq 1$ be integers. We have

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We say that MDS codes are **dense** within the set of k-dimensional codes in $\mathbb{F}_q^n$.

We study "density questions" in coding theory in:

Byrne, R., *Partition-Balanced Families of Codes and Asymptotoc Enumeration in Coding Theory*, J. Combinatorial Theory A, to appear.

The notion of density

Definition

Let $S \subseteq \mathbb{N}$ be an infinite set. Let $(\mathscr{F}_s \mid s \in S)$ be a sequence of finite non-empty sets indexed by S, and let $(\mathscr{F}'_s \mid s \in S)$ be a sequence of sets with $\mathscr{F}'_s \subseteq \mathscr{F}_s$ for all $s \in S$.

The **density function** $S \to \mathbb{Q}$ of $\mathscr{F}'_s$ in $\mathscr{F}_s$ is $\quad s \mapsto |\mathscr{F}'_s|/|\mathscr{F}_s|$.

If

$$\lim_{s \to +\infty} |\mathscr{F}'_s|/|\mathscr{F}_s| = \delta,$$

then $\mathscr{F}'_s$ has **density** δ in $\mathscr{F}_s$.

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Example

$S = \mathbb{N}_{\geq 1} \qquad \mathscr{F}_s = \{n \in \mathbb{N} \mid 1 \leq n \leq s\} \qquad \mathscr{F}'_s = \{p \in \mathbb{N} \mid p \leq s,\ p \text{ prime}\}$.

Then: $\qquad |\mathscr{F}'_s|/|\mathscr{F}_s| \to 0, \qquad |\mathscr{F}'_s|/|\mathscr{F}_s| \sim 1/\log(s)$

(Hadamard, de la Vallée-Poussin, 1896)

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- The k-dimensional MDS codes in $\mathbb{F}_q^n$ are in bijection with the non-zeros of a polynomial $p \in \mathbb{F}_q[z_1, \dots, z_N]$, where $N = k(n-k)$
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Density problems in coding theory

We study density problems in general:

- Ambient space: Hamming space, matrix rk-metric space, vector rk-metric space
- Various properties related to: minimum distance, covering radius, maximality

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Idea

Look at **families** of codes that exhibit regularity properties with respect to partitions of the ambient space $X \in \{\mathbb{F}_q^n,\ \mathbb{F}_q^{n\times m},\ \mathbb{F}_{q^m}^n\}$.

Definition

Let $\mathscr{P} = \{P_1, P_2, \dots, P_\ell\}$ be a partition of X.

A family $\mathscr{F}$ of codes in X is $\mathscr{P}$**-balanced** if for all $x \in X$ the number

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only depends on the class of x with respect to the partition $\mathscr{P}$.

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only depends on the class of x with respect to the partition $\mathscr{P}$.

We use $\mathscr{P}$-balanced families to estimate the number of codes with a certain property.

MRD vector rk-metric codes

Using the Schwartz-Zippel lemma:

Theorem (Neri-Trautmann-Randrianarisoa-Rosenthal, 2017)

For vector-rank-metric codes ($\mathbb{F}_{q^m}$-linear)

$$\frac{\#\text{ of } k\text{-dim MRD codes in } \mathbb{F}_{q^m}^n}{\#\text{ of } k\text{-dim codes in } \mathbb{F}_{q^m}^n} \geq q^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m}^{-1} \left(1 - \sum_{r=0}^{k} \begin{bmatrix} k \\ k-r \end{bmatrix}_q \begin{bmatrix} n-k \\ r \end{bmatrix}_q q^{r^2} q^{-m}\right)$$

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We can improve this bound as follows:

Theorem (Byrne-R.)

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Theorem (Byrne-R.)

Let $m \geq n \geq 2$ and let $1 \leq k \leq mn-1$ be integers.

- If m does not divide k, then there is no k-dimensional MRD code $\mathscr{C} \leq \mathbb{F}_q^{n \times m}$.
- If m divides k, then

$$\frac{\#\text{ of } k\text{-dim non-MRD codes in } \mathbb{F}_q^{n\times m}}{\#\text{ of } k\text{-dim codes in } \mathbb{F}_q^{n\times m}} \geq$$

$$q \begin{bmatrix} mn \\ k \end{bmatrix}^{-1} \left(\sum_{h=1}^{m(n-k)} \begin{bmatrix} t \\ h \end{bmatrix} \sum_{s=h}^{m(n-k)} \begin{bmatrix} m(n-k)-h \\ s-h \end{bmatrix} \begin{bmatrix} mn-s \\ mn-k \end{bmatrix} (-1)^{s-h} q^{\binom{s-h}{2}} \right) \cdot$$

$$\cdot \left(1 - \frac{\left(q^k-1\right)\left(q^{mn-k}-1\right)}{2\left(q^{mn}-q^{mn-k}\right)} \right).$$

The RHS goes to $1/2$ as $q \to +\infty$ and to $1/2(q/(q-1)-(q-1)^2)$ as $m \to +\infty$.

Non-density of MRD matrix codes

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Matrix MRD codes are not dense

Non-density was also shown by Antrobus/Gluesing-Luerssen with different methods.

Other results

We study:

- Density of codes that are **optimal** (MDS, MRD, MRD)
- Density of codes of bounded **minimum distance**
- Density of codes that meet the *redundancy bound* for their **covering radius**
- Density of matrix codes that meet the *initial set bound* for their covering radius
- Density of optimal codes within **maximal** codes (with respect to inclusion)
- ...

R., *Whitney numbers of combinatorial geometries and higher-weight Dowling lattices*, arXiv 1909.10249.

Codes with the Hamming metric and geometric lattices

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Example of question

How many codes $\mathscr{C} \leq \mathbb{F}_q^n$ are there of dimension k and $d_{\mathrm{H}}(\mathscr{C}) > d$?

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Theorem (Dowling 1971, Zaslavsky 1987)

Counting codes $\longleftarrow$ computing the ch. polynomials of certain geometric lattices.

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In particular, of **higher-weight Dowling lattices** (abbreviated **HWDLs**).

HWDLs

$\mathscr{H}(q,n,d)$ is a sublattice of the lattice of subspaces of $\mathbb{F}_q^n$.

Definition

$\mathscr{H}(q,n,d)$ consists of those subspaces of $\mathbb{F}_q^n$ that have a basis made of vectors of Hamming weight $\leq d$, ordered by inclusion.

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- Introduced by Dowling in 1971
- Studied by Dowling, Zaslavsky, Bonin, Kung, Brini, Games, ...
- To date, still very little is known about HWDLs
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Examples

- For $d=1$, $\mathscr{H}(q,n,d)$ is the Boolean algebra of subsets of $\{1,\dots,n\}$
- For $d=2$, $\mathscr{H}(q,n,d)$ is isomorphic to the q-analogue of the partition lattice (Dowling '73)

Studying these lattices is an old open problem in combinatorics.

HWDLs

Theorem (R., 2019)

The following are *equivalent*:

- (partial) knowledge of the number of codes with $d_{\mathrm{H}}(\mathscr{C}) > d$
- (partial) knowledge of the Whitney numbers of HWDL's

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The following are *equivalent*:

- (partial) knowledge of the number of codes with $d_{\mathrm{H}}(\mathscr{C}) > d$
- (partial) knowledge of the Whitney numbers of HWDL's

More precisely, let $\alpha_k(q,n,d) = \#\{\mathscr{C} \leq \mathbb{F}_q^n \mid \dim(C) = k,\ d_{\mathrm{H}}(\mathscr{C}) > d\}$. Then

$$\alpha_k(q,n,d) = \sum_{i=0}^{k} w_i(q,n,d) \begin{bmatrix} n-i \\ k-i \end{bmatrix}_q \quad \text{for } 0 \leq k \leq n$$

$$w_i(q,n,d) = \sum_{k=0}^{i} \alpha_k(q,n,d) \begin{bmatrix} n-k \\ i-k \end{bmatrix}_q (-1)^{i-k} q^{\binom{i-k}{2}} \quad \text{for } 0 \leq i \leq n$$

Recall: the i-th Whitney number of $\mathscr{L} = \mathscr{H}(q,n,d)$ is

$$w_i(q,n,d) = \sum_{\mathrm{rk}(x)=i} \mu_{\mathscr{L}}(0,x)$$

HWDLs

Theorem (R., 2019)

For all $n \geq 9$ we have

$$
\begin{aligned}
-w_3(2,n,3) \;=\; & \sum_{1\leq \ell_1<\ell_2<\ell_3\leq n-2} \left(\prod_{j=1}^{3} \binom{n-\ell_j-9+3j)}{2} \right) + 8\binom{n}{3} \sum_{s=3}^{8} \binom{n-3}{n-s} (-1)^{s-3} \\
& + 106\binom{n}{4} \sum_{s=4}^{8} \binom{n-4}{n-s} (-1)^{s-4} + 820\binom{n}{5} \sum_{s=5}^{8} \binom{n-5}{n-s} (-1)^{s-5} \\
& + 4565\binom{n}{6} \sum_{s=6}^{8} \binom{n-6}{n-s} (-1)^{s-6} \\
& + 19810\binom{n}{8} \sum_{s=7}^{8} \binom{n-7}{n-s} (-1)^{s-7} + 70728\binom{n}{8}.
\end{aligned}
$$

HWDLs

Theorem (R., 2019)

Let $n \geq 6$. We have

$$\begin{aligned} w_2(q,n,3) &= \frac{1}{72}q^4n^6 - \frac{1}{12}q^4n^5 + \frac{1}{18}q^4n^4 + \frac{1}{2}q^4n^3 - \frac{77}{72}q^4n^2 + \frac{7}{12}q^4n - \frac{1}{18}q^3n^6 \\ &+ \frac{5}{12}q^3n^5 - \frac{49}{72}q^3n^4 - \frac{7}{6}q^3n^3 + \frac{269}{72}q^3n^2 - \frac{9}{4}q^3n + \frac{1}{12}q^2n^6 - \frac{3}{4}q^2n^5 \\ &+ 2q^2n^4 - \frac{7}{12}q^2n^3 - \frac{43}{12}q^2n^2 + \frac{17}{6}q^2n - \frac{1}{18}qn^6 + \frac{7}{12}qn^5 - \frac{157}{72}qn^4 \\ &+ \frac{19}{6}qn^3 - \frac{55}{72}qn^2 - \frac{3}{4}qn + \frac{1}{72}n^6 - \frac{1}{6}n^5 + \frac{29}{36}n^4 - \frac{23}{12}n^3 + \frac{157}{72}n^2 - \frac{11}{12}n. \end{aligned}$$

HWDLs

Theorem (R., 2019)

For all integers $n \geq d \geq 2$ and any prime power q,

$$w_2(q,n,d) = (q^{n-1}-1)\sum_{j=1}^{d}\binom{n}{j}(q-1)^{j-2} - \sum_{1\leq \ell_1<\ell_2\leq n}\Bigg[q^{n-\ell_1-1}\left(\sum_{j=0}^{d-1}\binom{n-\ell_2}{j}(q-1)^j\right)$$

$$+\sum_{j=d}^{n-\ell_2}\sum_{h=0}^{d-1}\binom{n-\ell_2}{j}\binom{n-\ell_1-1}{h}(q-1)^{j+h}$$

$$+\sum_{s=d}^{n-\ell_2}\sum_{t=0}^{d-2}\binom{n-\ell_2}{s}\binom{n-\ell_1-1-s}{t}(q-1)^{s+t}\sum_{v=d-t}^{s}\gamma_q(s,s-d+t+2,v)\Bigg],$$

where the $\gamma_a(b,c,v)$'s are the *agreement numbers*.

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For all integers $n \geq d \geq 2$ and any prime power q,

$$w_2(q,n,d) = (q^{n-1}-1)\sum_{j=1}^{d}\binom{n}{j}(q-1)^{j-2} - \sum_{1\leq \ell_1<\ell_2\leq n}\Bigg[q^{n-\ell_1-1}\left(\sum_{j=0}^{d-1}\binom{n-\ell_2}{j}(q-1)^j\right)$$
$$+\sum_{j=d}^{n-\ell_2}\sum_{h=0}^{d-1}\binom{n-\ell_2}{j}\binom{n-\ell_1-1}{h}(q-1)^{j+h}$$
$$+\sum_{s=d}^{n-\ell_2}\sum_{t=0}^{d-2}\binom{n-\ell_2}{s}\binom{n-\ell_1-1-s}{t}(q-1)^{s+t}\sum_{v=d-t}^{s}\gamma_q(s,s-d+t+2,v)\Bigg],$$

where the $\gamma_a(b,c,v)$'s are the *agreement numbers*.

$\gamma_a(b,c,v)$ is a polynomial in a (for any b, c and v) whose coefficients are expressions involving the Bernoulli numbers:

$$\frac{x}{e^x-1} = \sum_{n=0}^{+\infty} B_n \frac{x^n}{n!}.$$

$\rightarrow$ **polynomiality** in q of $w_2(q,n,d)$ (R. 2019)

Back to density results

Theorem (Folklore)

Let $n \geq k \geq 1$ be integers. We have

$$\lim_{q \to +\infty} \frac{\#\text{ of } k\text{-dim MDS codes in } \mathbb{F}_q^n}{\#\text{ of } k\text{-dim codes in } \mathbb{F}_q^n} = 1.$$

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Thank you very much!