Matrix Codes and Rook Theory

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joint work with H. Gluesing-Luerssen

A classical result in coding theory:

Theorem (MacWilliams)

Let $\mathscr{C} \leq \mathbb{F}_q^n$ be a code with the Hamming metric. Then for all $0 \leq j \leq n$ we have

$$W_j^{\mathrm{H}}(\mathscr{C}^\perp) = \sum_{i=0}^{n} \sum_{\ell=0}^{j} (-1)^\ell (q-1)^{j-\ell} \binom{i}{\ell} \binom{n-i}{j-\ell} W_i^{\mathrm{H}}(\mathscr{C}).$$

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Generalizations of this result have been extensively studied in various contexts:

- association schemes
- finite abelian groups
- posets/lattices

Group Characters

Definition

Let $(G,+)$ be a finite abelian group. The character group of G is

$$\widehat{G} = \{\text{group homomorphisms } \chi : G \to \mathbb{C}^*\}$$

endowed with point-wise multiplication:

$$\chi_1 \cdot \chi_2 \ (g) = \chi_1(g) \cdot \chi_2(g) \quad \text{for all } g \in G.$$

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We focus on a special situation:

- $(G,+) = (V,+)$ is the additive group of a finite-dimensional linear space over $\mathbb{F}_q$
- V is endowed with a given scalar product $\langle \cdot \ \cdot \rangle$

Remark

$(\widehat{V}, \cdot)$ has a natural structure of $\mathbb{F}_q$-linear space via

$$a\chi(v) = \chi(av), \qquad a \in \mathbb{F}_q, \ v \in V.$$

Moreover, $\dim(V) = \dim(\widehat{V})$.

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Remark

$\langle \cdot \, \cdot \rangle$ can be used to identify the spaces $(V,+)$ and $(\widehat{V},\cdot)$ as follows.

Fix a non-trivial character $\xi : \mathbb{F}_q \to \mathbb{C}^*$ and let

$$\psi_\xi : V \to \widehat{V}, \qquad \psi_\xi(v)(w) = \xi(\langle v, w \rangle) \quad \text{for all } v, w \in V.$$

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Theorem (Folklore)

ψ_ξ is an $\mathbb{F}_q$-isomorphism of linear spaces whenever ξ is non-trivial.

Different choices of ξ give different identifications. However, all the objects we are interested in will not depend on the choice of ξ.

Partitions

A partition $\mathscr{P} = \{P_i\}_{i \in I}$ of V is **invariant** if $aP_i = P_i$ for all $i \in I$ and $a \in \mathbb{F}_q \setminus \{0\}$.

Example

Partitioning the elements of $\mathbb{F}_q^n$ according to their Hamming weight yields $\mathscr{P}^{\mathrm{H}}$.

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The **dual** of $\mathscr{P}$ is the partition $\widehat{\mathscr{P}}$ of V defined by the equivalence relation

$$w \sim w' \iff \sum_{v \in P_i} \psi_\xi(v)(w) = \sum_{v \in P_i} \psi_\xi(v)(w') \quad \text{for all } i \in I.$$

(recall: $\psi_\xi : (V, +) \to (\widehat{V}, \cdot)$ $\mathbb{F}_q$-isomorphism).

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I am using ξ to define $\widehat{\mathscr{P}}$. However,

Proposition

$\widehat{\mathscr{P}}$ does not depend on ξ, if $\mathscr{P}$ is invariant.

Partitions

DATA:

- V an $\mathbb{F}_q$-space of finite dimension
- $\langle \cdot \ \cdot \rangle$ a scalar product on V
- $\mathscr{P} = \{P_i\}_{i \in I}$ an invariant partition of V

CONSTRUCTION: the dual partition $\widehat{\mathscr{P}} = \{Q_j\}_{j \in J}$ of V (which is invariant as well)

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Definition

A **code** is an $\mathbb{F}_q$-subspace of V. Its **dual** is

$$\mathscr{C}^\perp = \{w \in V \mid \langle v, w \rangle = 0 \text{ for all } v \in \mathscr{C}\} \leq V.$$

Define:

- the $\mathscr{P}$**-distribution** of $\mathscr{C}$: $\mathscr{P}(\mathscr{C}, i) = |\mathscr{C} \cap P_i|, \quad i \in I.$
- the $\widehat{\mathscr{P}}$**-distribution** of $\mathscr{C}^\perp$: $\widehat{\mathscr{P}}(\mathscr{C}^\perp, j) = |\mathscr{C}^\perp \cap Q_j|, \quad j \in J.$

Under certain conditions, MacWilliams-type identities hold for the $\mathscr{P}$- and $\widehat{\mathscr{P}}$-partition.

MacWilliams-type Identities

We say that $\mathscr{P}$ is **Fourier-reflexive** if $|\mathscr{P}| = |\widehat{\mathscr{P}}|$ and **self-dual** if $\widehat{\mathscr{P}} = \mathscr{P}$.

(self-dual $\Longrightarrow$ Fourier-reflexive)

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Theorem (Generalized MacWilliams Identities)

Let $\mathscr{P} = \{P_i\}_{i \in I}$ be invariant and Fourier-reflexive.

Let $\widehat{\mathscr{P}} = \{Q_j\}_{j \in J}$. Let $\mathscr{C} \leq V$ be a code. We have

$$\widehat{\mathscr{P}}(\mathscr{C}^\perp, j) = \frac{1}{|\mathscr{C}|} \sum_{i \in I} K(\mathscr{P}; i, j) \cdot \mathscr{P}(\mathscr{C}, i),$$

where $K(\mathscr{P}; i, j)$ are suitable numbers called **Krawtchouk coefficients**. Moreover, the matrix of the $K(\mathscr{P}; i, j)$ is of size $|I| \times |J| = |I| \times |I|$ and invertible.

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Definition

$$K(\mathscr{P}; i, j) = \sum_{w \in Q_j} \psi_{\xi}(w)(v), \quad \text{where } v \text{ is any vector in } P_i.$$

Again, this does not depend on ξ for invariant partitions.

Problems

Given V with $\langle \cdot \ \cdot \rangle$,

- Construct Fourier-reflexive partitions $\mathscr{P}$
- Describe $\widehat{\mathscr{P}}$ and decide if $\widehat{\mathscr{P}} = \mathscr{P}$ (self-duality)
- Compute $K(\mathscr{P}; i, j)$

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Theorem (Delsarte)

The rank partition on $\mathbb{F}_q^{n \times m}$ is self-dual of size $m+1$. Moreover,

$$K(\mathscr{P}^{\mathrm{rk}}; i, j) = \sum_{\ell=0}^{m} (-1)^{j-\ell} q^{n\ell + \binom{j-\ell}{2}} \begin{bmatrix} m-\ell \\ m-j \end{bmatrix}_q \begin{bmatrix} m-i \\ \ell \end{bmatrix}_q .$$

We concentrate on the matrix space $\mathbb{F}_q^{n \times m}$ with $n \geq m$ endowed with the **trace product**:

$$\langle M, N \rangle = \mathrm{Tr}(MN^t).$$

Other partitions

We study:

- the row-space partition $\mathscr{P}^{\text{rs}}$
- the pivot partition $\mathscr{P}^{\text{piv}}$

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Results (Gluesing-Luerssen, R.):

- $\mathscr{P}^{\text{rs}}$ is self-dual
- explicit formula for the Krawtchouk coefficients of $\mathscr{P}^{\text{rs}}$
- the pivot partition $\mathscr{P}^{\text{piv}}$ is Fourier-reflexive (not self-dual)
- connection between the Krawtchouk coefficients of $\mathscr{P}^{\text{piv}}$ and rook theory
- notions of extremality from $\mathscr{P}^{\text{rs}}$ and $\mathscr{P}^{\text{piv}}$, and properties of extremal codes
- MacWilliams extension theorem fails for these partitions

The pivot partition

$\mathscr{P}^{\mathrm{piv}}$ partitions the elements of $\mathbb{F}_q^{n\times m}$ according to the pivot indices in the RRE form.
We have:

$$|\mathscr{P}^{\mathrm{piv}}| = \sum_{r=0}^{m} \binom{m}{r} = 2^m.$$

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Example:

$$M = \begin{pmatrix} 1 & \bullet & 0 & 0 & \bullet \\ 0 & 0 & 1 & 0 & \bullet \\ 0 & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{piv}(M) = (1,3,4).$$

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Notation

Let $\quad \Pi = \{(j_1,\dots,j_r) \mid 1 \leq r \leq m,\ \ 1 \leq j_1 < j_2 < \cdots < j_r \leq m\} \cup \{()\}.$

Then

$$\mathscr{P}^{\mathrm{piv}} = (P_\lambda)_{\lambda \in \Pi}.$$

We treat the elements of Π as sets or as lists, depending on what is more convenient.

Theorem (Gluesing-Luerssen, R.)

$\mathscr{P}^{\mathsf{piv}}$ is Fourier-reflexive, but not self-dual $\;(\widehat{\mathscr{P}^{\mathsf{piv}}} \neq \mathscr{P}^{\mathsf{piv}})$.

How does $\widehat{\mathscr{P}^{\mathsf{piv}}}$ look like?

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$\widehat{\mathscr{P}^{\mathrm{piv}}} = \mathscr{P}^{\mathrm{rpiv}}$, the reverse pivot partition.

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Computing the Krawtchouk coefficients is a different story...

The pivot partition

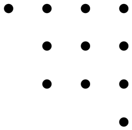
Definition

A **Ferrers diagram** is a subset $\mathcal{F} \subseteq [n] \times [m]$ that satisfies the following:

1. if $(i,j) \in \mathcal{F}$ and $j < m$, then $(i,j+1) \in \mathcal{F}$ (right aligned),
2. if $(i,j) \in \mathcal{F}$ and $i > 1$, then $(i-1,j) \in \mathcal{F}$ (top aligned).

We represent a Ferrers diagram by its column lengths, $\mathcal{F} = [c_1, \ldots, c_m]$.

E.g.

$\mathcal{F} =$  $= [1,3,3,4]$

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We denote by $\mathbb{F}_q[\mathcal{F}]$ the space of matrices supported on $\mathcal{F}$, and let

$$P_r(\mathcal{F}) := \{M \in \mathbb{F}_q[\mathcal{F}] \mid \mathrm{rk}(M) = r\}.$$

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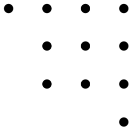

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We can express the Krawtchouk coefficients of $\mathscr{P}^{\mathrm{piv}}$ in terms of $P_r(\mathscr{F})$, for certain r and for a suitable diagram $\mathscr{F}$.

The pivot partition

Using combinatorial tools (*regular support functions*):

Theorem (Gluesing-Luerssen, R.)

Let $\lambda, \mu \in \Pi$. Set

$$\sigma = [m] \setminus \mu, \qquad \lambda \cap \sigma = (\lambda_{\alpha_1}, \dots, \lambda_{\alpha_x}), \qquad \mu \setminus \lambda = (\mu_{\beta_1}, \dots, \mu_{\beta_y}).$$

Furthermore, set

$$z_j = |\{i \in [x] \mid \lambda_{\alpha_i} < \mu_{\beta_j}\}| \text{ for } j \in [y], \qquad \mathscr{F} = [z_1, \dots, z_y].$$

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Then

$$K(\mathscr{P}^{\text{piv}}; \lambda, \mu) = \sum_{t=0}^{m} (-1)^{|\lambda|-t} q^{nt+\binom{|\lambda|-t}{2}} \sum_{r=0}^{|\lambda\cap\sigma|} P_r(\mathscr{F}) \begin{bmatrix} |\lambda \cap \sigma| - r \\ t \end{bmatrix}_q .$$

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Therefore, $K(\mathscr{P}^{\text{piv}}; \lambda, \mu)$ can be expressed in terms of the rank-distribution of $\mathbb{F}_q(\mathscr{F})$ for a suitable $\mathscr{F}$ $\rightarrow$ **rook theory**

Definition

The ***q*-rook polynomial** associated with $\mathscr{F}$ and $r \geq 0$ is

$$R_r(\mathscr{F}) = \sum_{C \in \mathrm{NAR}_r(\mathscr{F})} q^{\mathrm{inv}(C,\mathscr{F})} \in \mathbb{Z}[q],$$

where:

- $\mathrm{NAR}_r(\mathscr{F})$ is the set of all placements of r non-attacking rooks on $\mathscr{F}$ (non-attacking means that no two rooks are in the same column, and no two are in the same row)
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Theorem (Haglund)

For any Ferrers diagram $\mathscr{F}$ and any $r \geq 0$ we have

$$P_r(\mathscr{F}) = (q-1)^r \, q^{|\mathscr{F}|-r} \, R_r(\mathscr{F})_{|q^{-1}}$$

in the ring $\mathbb{Z}[q, q^{-1}]$.

Natural task: find an explicit expression for $R_r(\mathscr{F})$.

An explicit formula for $R_r(\mathscr{F})$:

Theorem (Gluesing-Luerssen, R.)

Let $\mathscr{F} = [c_1, \dots, c_m]$ be an $n \times m$-Ferrers diagram. For $k \in [m]$ define $a_k = c_k - k + 1$.

For $j \in [m]$ let $\sigma_j \in \mathbb{Q}[x_1, \dots, x_m]$ be the j^{th} elementary symmetric polynomial in m indeterminates ($\sigma_0 = 1$, ..., $\sigma_m = x_1 \cdots x_m$).

Then

$$R_r(q) = \frac{q^{\binom{r+1}{2} - rm + \text{area}(\mathscr{F})} (-1)^{m-r}}{(1-q)^r \prod_{k=1}^{m-r}(1-q^k)} \sum_{t=m-r}^{m} (-1)^t \sigma_{m-t}(q^{-a_1}, \dots, q^{-a_m}) \prod_{j=0}^{m-r-1} (1-q^{t-j}).$$

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Combining this with Haglund's theorem we find an explicit expression for $P_r(\mathscr{F})$.

Proof is technical.

A different approach: compute $P_r(\mathscr{F})$ directly. Notation: $\mathscr{F} = [c_1, \dots, c_m]$.

Theorem (Gluesing-Luerssen, R.)

$$P_r(\mathscr{F}) = \sum_{1 \leq i_1 < \cdots < i_r \leq m} q^{rm - \sum_{j=1}^{r} i_j} \prod_{j=1}^{r} (q^{c_{i_j} - j + 1} - 1).$$

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Proof is short.

But inverting Haglund's theorem we also find a simple explicit formula for $R_r(\mathscr{F})$!

Corollary (Gluesing-Luerssen, R.)

$$R_r(\mathscr{F}) = \frac{q^{\sum_{j=1}^m c_j - rm} \displaystyle\sum_{1 \le i_1 < \cdots < i_r \le m} \prod_{j=1}^r (q^{i_j + j - c_{i_j} - 1} - q^{i_j})}{(1-q)^r}.$$

We can use these results to derive an explicit formula for the q-Stirling numbers of the second kind. The latter are defined via the recursion

$$S_{m+1,r} = q^{r-1} S_{m,r-1} + \frac{q^r - 1}{q-1} S_{m,r}$$

with initial conditions $S_{0,0}(q) = 1$ and $S_{m,r}(q) = 0$ for $r < 0$ or $r > m$.

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$$S_{m+1,m+1-r} = R_r(\mathscr{F}),$$

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Thank you very much!